Received: *June 12, 1998*, Accepted: *July 14, 1998*

Baryons and branes in anti de Sitter space

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Abstract: In the mapping from four-dimensional gauge theories to string theory in AdS space, many features of gauge theory can be described by branes wrapped in different ways on $\mathbf{S}^5$, $\mathbf{RP}^5$, or subspaces therefore. These include a baryon vertex coupling N external charges in the fundamental representation of $SU(N)$, a bound state of k gluons in $SO(2k)$ gauge theory, strings coupled to external charges in the spinor representation of the gauge group, and domain walls across which the low energy gauge group changes.

Keywords: 1/N Expansion, String Duality, Black Holes in String Theory.

JHEP07(1998)006

Contents

1. Introduction

It has been argued [1] that four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory, with gauge group $SU(N)$, is equivalent to Type IIB superstring theory on $AdS_5 \times \mathbf{S}^5$ (where AdS_5 is five-dimensional Anti-de Sitter space, and there are N units of five-form flux on the five-sphere $\mathbf{S}^5$). This fascinating subject has been developed in many directions. Precise recipes for computing correlation functions of local operators in this framework have been presented in [2, 3]; there likewise are precise recipes for computing correlation functions involving Wilson loop operators [4, 5]. A similar treatment can be given for models with reduced supersymmetry that are obtained by an orbifolding operation in which $\mathbf{S}^5$ is replaced by $\mathbf{S}^5/\Gamma$, with Γ a finite group [6]. Of particular importance in the present paper, the gauge group $SU(N)$ can be replaced by $SO(N)$ or $Sp(N/2)$ by an orientifolding operation in which $\mathbf{S}^5$ is replaced by $\mathbf{RP}^5 = \mathbf{S}^5/\mathbf{Z}_2$ (analogous orientifolds in eleven dimensions were discussed in [7]; such orbifolds in ten dimensions were discussed in [8] and related explicitly to supergravity in [9]).

The present paper began with the following question. Since the $AdS_5 \times \mathbf{S}^5$ theory encodes an $SU(N)$ gauge theory, rather than $U(N)$, should there not be a baryon vertex? In other words, should there not be finite energy configurations with N external quarks, roughly in parallel with the external quark-antiquark configurations studied in [4, 5]?

In section 2, we will construct such a baryon vertex. It has a simple interpretation; it is obtained by wrapping a fivebrane over $\mathbf{S}^5$!

Finding a baryonic vertex in the $\mathcal{N}=4$ theory does not mean that that theory has baryonic particles, or operators. Baryonic particles would appear in a theory that has dynamical quark fields (that is, fields transforming in the fundamental representation of $SU(N)$); in their absence, we get only a baryonic vertex, a gauge-invariant coupling of N external charges. Introducing dynamical quark fields would require breaking some supersymmetry. As an alternative route to studying an object somewhat like a baryonic particle, we can replace the gauge group $SU(N)$ with $SO(N)$, without breaking any supersymmetry. Take N to be an even number, $N=2k$. $SO(2k)$ gauge theory admits a gauge-invariant configuration of $k=N/2$ gauge bosons. In fact, if Φ_{ab}, $a,b=1,\dots,2k$ is an antisymmetric second rank tensor, transforming in the adjoint representation of $SO(2k)$, then the "Pfaffian" $\mathrm{Pf}(\Phi)=(1/k!)\epsilon^{a_1a_2\dots a_{2k}}\Phi_{a_1a_2}\dots\Phi_{a_{2k-1}a_{2k}}$ is an irreducible gauge-invariant polynomial of order $N/2$. We will call such operators Pfaffian operators, and the particles they create Pfaffian particles.

The Pfaffian particle has long presented a puzzle for the general understanding of the $1/N$ expansion of gauge theories.[1] It is suspected [12] that Yang-Mills theory with $SU(N)$, $SO(N)$, or $Sp(N)$ gauge group has a large N limit as a closed string theory, with an effective string coupling constant $\lambda\sim 1/N$. If elementary quarks are added, it is believed that there are also open strings (describing mesons), with an open string coupling constant $\lambda'\sim 1/\sqrt{N}$. Baryons of an $SU(N)$ theory with dynamical quarks are N-quark bound states, which one would expect to have masses of order N. As $N\sim 1/(\lambda')^2$, such states can be interpreted as solitons in the open string sector [13].

The Pfaffian particle of an $SO(N)$ gauge theory without quarks, being a bound state of $N/2$ gluons, is intuitively expected to have a mass of order N. This particle cannot be interpreted as a soliton because, in terms of the closed string coupling λ, its mass is of order $1/\lambda$, not $1/\lambda^2$. Given the modern understanding of D-branes [14], one might wonder if the Pfaffian particle of $SO(N)$ is a D-brane.[2]

This question can be addressed by comparing $SO(N)$ gauge theory to string theory on $AdS_5\times\mathbf{RP}^5$. With this aim, we describe in section 3 the basic rules for wrapping branes on $\mathbf{RP}^5$. We explain the "discrete torsion" that distinguishes the $SO(2k)$, $SO(2k+1)$, and $Sp(k)$ theories, and its relation to Montonen-Olive duality. We also describe some important topological restrictions on brane wrapping.

We then go on in section 4 to discuss the gauge theory interpretation of various types of wrapped branes. In section 4.1, we show that in $SO(2k)$ gauge theory, interpreted

[1] Roughly such questions about N dependence motivated early pre-D-brane work (see for example [10]) and early conjectures [11] about the structure of nonperturbative corrections in string theory.

[2] Likewise, might the baryon of an $SU(N)$ theory with dynamical quarks be a D-brane rather than an open string soliton? This question is not quite well-defined because the $SU(N)$ theory with quarks has open strings. In theories with open strings, D-branes and open string solitons can describe the same objects, as in the case of Type I fivebranes/Yang-Mills instantons [15]. Which description is more useful can depend on the circumstances.

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in terms of $AdS_5 \times \mathbf{RP}^5$, there is a natural candidate for the Pfaffian – a threebrane wrapped on an $\mathbf{RP}^3$ subspace of $\mathbf{RP}^5$.[3] We show that a threebrane cannot be so wrapped in $SO(2k+1)$ or $Sp(k)$ – as expected, since these groups do not have gauge-invariant Pfaffians. For $SO(2k+1)$, we show that one can have a wrapped threebrane with an attached string; we explain the meaning of this object in gauge theory.

The rest of section 4 is devoted to other types of brane wrapping. In section 4.2, we consider strings obtained by wrapping a fivebrane on an $\mathbf{RP}^4$ subspace of $\mathbf{RP}^5$. We argue that such strings can be used to compute, in $SO(2k)$ or $SO(2k+1)$ gauge theory, the expectation values of Wilson lines in the spinor representation of the gauge group. (We also show that such wrapped branes are not possible in $Sp(k)$ gauge theory – as expected since $Sp(k)$ has no spinor representation.) In section 4.3, we consider three-branes on AdS_5 – both the unwrapped Type IIB threebrane and additional threebranes made by wrapping a fivebrane on $\mathbf{RP}^2 \subset \mathbf{RP}^5$. We argue that such threebranes behave as domain walls, with the property that the gauge group jumps when one crosses one. In section 4.4, we consider the baryon vertex – the antisymmetric coupling of N external quarks – in the orthogonal and symplectic gauge theories. Finally, in section 4.5, we reexamine the -1-branes of AdS space – which were recently discussed as Yang-Mills instantons [17].

2. The baryon vertex in $SU(N)$ gauge theory

First we recall how external quarks in $\mathcal{N}=4$ super Yang-Mills are described in terms of Type IIB on $AdS_5 \times \mathbf{S}^5$ [4, 5]. With Lorentz signature, the boundary of AdS_5 is $\mathbf{S}^3 \times \mathbf{R}$, where $\mathbf{R}$ is the "time" direction; $\mathbf{S}^3 \times \mathbf{R}$ is a universal cover of the conformal compactification of Minkowski space. External quarks are regarded as endpoints of strings in AdS space. Thus, to compute the energy for a time-independent configuration with an external quark at a point $x \in \mathbf{S}^3$ and an external antiquark at $y \in \mathbf{S}^3$, one considers configurations in which a string inside the AdS space connects the boundary points x and y.

The strings in question are elementary Type IIB superstrings if the external charges are electric charges, that is particles in the fundamental representation of the gauge group $SU(N)$. External monopoles would be boundaries of D-strings, and external charges of type (p,q) are boundaries of (p,q) strings.

As promised in the introduction, we now want to find a "baryon" vertex connecting N external quarks, with their color wave functions combined together by an N^{th} order antisymmetric tensor of $SU(N)$. For this, we place external quarks at boundary points $x_1, x_2, \ldots, x_N$. We consider a configuration in which each of the boundary points is the endpoint of an elementary superstring in $AdS_5 \times \mathbf{S}^5$, with all N strings oriented in the

[3] As explained at the end of section 4.1, the wrapped threebrane is really related to a Pfaffian operator of the boundary theory rather than a Pfaffian particle; it would be related to a Pfaffian particle after a suitable perturbation that breaks conformal invariance.

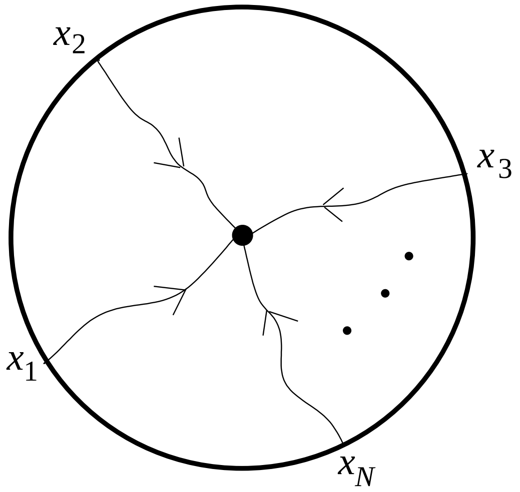

Figure 1: N elementary strings attached to points $x_1, x_2, \ldots, x_N$ on the boundary of AdS space and joining at a baryon vertex in the interior.

same way. We want, as in figure 1, to find a "baryon vertex," where these N strings can somehow terminate in the interior of $AdS_5 \times \mathbf{S}^5$.

We claim that the baryon vertex is simply a wrapped fivebrane – i.e., a fivebrane whose world-volume is $\mathbf{S}^5 \times \mathbf{R}$, with $\mathbf{R}$ a timelike curve in AdS_5. With a suitable choice of "time" coordinate in AdS_5, this is the worldvolume of a static fivebrane wrapped on $\mathbf{S}^5 \times Q$, where Q is a point in a time zero slice of AdS_5. Assuming that the strings that are to be joined at the baryon vertex are elementary strings, we will build the baryon vertex from a D-fivebrane; the baryon vertex for N external charges of the same type (p,q) would be similarly made by wrapping a (p,q) fivebrane on $\mathbf{S}^5$. For definiteness, in what follows, we consider the case of elementary strings and D-fivebranes.

The reason that the wrapped fivebrane is a baryon vertex is the following. In Type IIB superstring theory, there is a self-dual five-form field G_5. The $AdS_5 \times \mathbf{S}^5$ compactification which is related to $SU(N)$ gauge theory has N units of five-form flux on $\mathbf{S}^5$:

$$\int_{\mathbf{S}^5} \frac{G_5}{2\pi} = N\,. \tag{2.1}$$

On the D-fivebrane worldvolume, there is a $U(1)$ gauge field a. It couples to G_5 by a coupling

$$\int_{\mathbf{S}^5 \times \mathbf{R}} a \wedge \frac{G_5}{2\pi}\,. \tag{2.2}$$

Because of this coupling and (2.1), the G_5 field contributes N units of a-charge. Since the total charge of a $U(1)$ gauge field must vanish in a closed universe, there must be $-N$ units of charge from some other source.

Such a source is an elementary string ending on the fivebrane. As in [18], the endpoint of an elementary string that ends on the fivebrane is electrically charged with respect to the a-field, with a charge that is $+1$ or -1 depending on the orientations of the string and fivebrane. To cancel the G_5 contribution to the a-charge, we need N

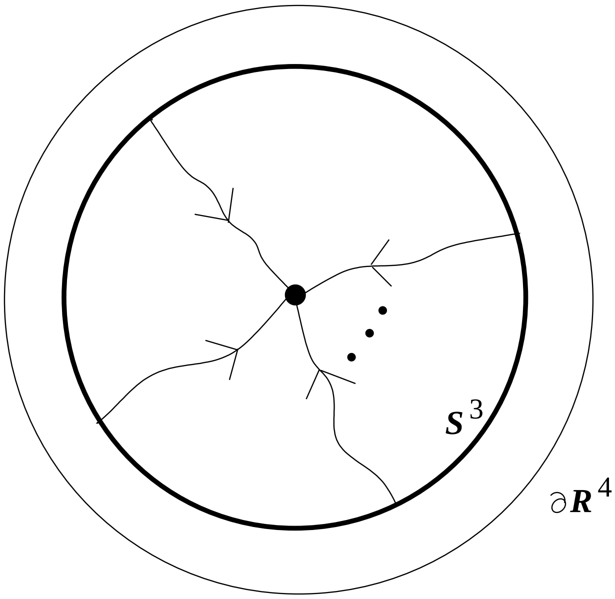

Figure 2: This differs from figure 1 in that now a threebrane wraps over a large three-sphere in AdS space, and the N elementary strings terminate on the threebrane rather than the boundary. In the figure, the boundary of AdS space (a spatial section of which is topologically $\mathbf{R}^4$) is denoted $\partial\mathbf{R}^4$, and the spatial section of the threebrane is denoted $\mathbf{S}^3$.

strings, all oriented in the same way, ending on the fivebrane. The fivebrane is thus a baryon or antibaryon vertex, depending on its orientation.

We have thus provided evidence that the gauge theory on $\mathbf{S}^3 \times \mathbf{R}$ that is dual to Type IIB on $AdS_5 \times \mathbf{S}^5$ has the property that it is possible to form a gauge-invariant combination of N quarks, that is, of N particles in the fundamental representation of the gauge group. This is in agreement with the fact that the gauge group is believed to be $SU(N)$ (and not $U(N)$, for example). But group theory predicts more. In $SU(N)$ gauge theory, the gauge-invariant combination of N elementary quarks is completely antisymmetric. How do we see the antisymmetry in $AdS_5 \times \mathbf{S}^5$ string theory?

The antisymmetry means that elementary strings that connect a D-fivebrane to the boundary of AdS space behave as fermions. Since the boundary of AdS is only at a finite distance from a conformal point of view, to describe a string that stretches to the boundary of AdS space, we need a boundary condition at spatial infinity. We will now show that with certain natural boundary conditions, the strings stretching to the boundary in fact behave as fermions, giving the antisymmetry of the baryon vertex.

Natural boundary conditions are suggested by the original argument [4, 5] for regarding elementary quarks on $\mathbf{S}^3 \times \mathbf{R}$ as boundaries of strings in AdS space. In this argument, strings with an endpoint at infinity are regarded as limiting cases of strings with an endpoint on a three-brane that is, in a suitable sense, near to infinity. Thus, a time zero section of AdS_5 is a copy of $\mathbf{R}^4$, so a time zero section of $AdS_5 \times \mathbf{S}^5$ is a copy of $\mathbf{R}^4 \times \mathbf{S}^5$. We consider (figure 2) a static threebrane whose world-volume, at time zero, is a subspace of $\mathbf{R}^4 \times \mathbf{S}^5$ of the form $\mathbf{S}^3 \times R$, with $\mathbf{S}^3$ a sufficiently large three-sphere near infinity in $\mathbf{R}^4$ and R a point in $\mathbf{S}^5$. Our N strings thus connect the threebrane on $\mathbf{S}^3 \times R$ with the fivebrane on $Q \times \mathbf{S}^5$.

Now, the three-manifolds $\mathbf{S}^3 \times R$ and $Q \times \mathbf{S}^5$ are "linked" in the nine-manifold $\mathbf{R}^4 \times \mathbf{S}^5$ – they have linking number ± 1, depending on orientation. It has been seen in [19] that such linked branes, under certain conditions, are connected by elementary strings; the argument is a close cousin of the argument by which we deduced above that the wrapped fivebrane in $AdS_5 \times \mathbf{S}^5$ is a baryon vertex. As part of one explanation of the linking phenomenon, it has been observed [20] that the ground state of a string stretching between linked D-branes is fermionic and nondegenerate.[4] This is just right to make the baryon vertex antisymmetric, so if the boundary conditions are the ones implied in [4, 5], the baryon vertex is completely antisymmetric under permutation of the N strings.

Let us estimate the energy of the baryon vertex in the 't Hooft limit (the string coupling λ going to zero, and N to infinity, with λN fixed). Since the D-brane tension is of order $1/\lambda$, that is of order N, and the volume of $\mathbf{S}^5$ remains finite for $N \to \infty$, the baryon vertex for N external electric charges has an energy of order N, as expected for baryons in the large N limit of QCD [13]. For N external magnetic charges, the energy of the baryon vertex is controlled by the tension of an NS fivebrane, and has an extra factor of $1/\lambda$ or N; this factor seems natural, as magnetic charges are boundaries of D-strings, whose tension is $1/\lambda$ or N times the tension of the elementary strings. In each case, the energy of the baryon vertex is comparable to the energy of the N strings that are attached to it, and hence a dynamical study of baryonic states would involve balancing these two energies.

3. Orthogonal and symplectic gauge groups

3.1. The $\mathbf{RP}^5$ orientifold

By considering N parallel threebranes in $\mathbf{R}^{10}$, one gets a $U(N)$ gauge theory. Upon consideration of the associated supergravity solution and its near-horizon geometry, one gets the $AdS_5 \times \mathbf{S}^5$ description of $SU(N)$ gauge theory.

To obtain $SO(N)$ or (for even N) $Sp(N/2)$ gauge symmetry, one can instead consider N parallel threebranes at an orientifold threeplane. In other words, one replaces $\mathbf{R}^{10}$ by $\mathbf{R}^4 \times (\mathbf{R}^6/\mathbf{Z}_2)$ (with the $\mathbf{Z}_2$ acting by sign change on all six coordinates of $\mathbf{R}^6$) and places the threebranes at the singularity in $\mathbf{R}^6/\mathbf{Z}_2$. The orientifolding operation

[4] Instead of linked D-branes, one can take any transverse D-branes whose total dimensions add up to eight; the picture looks the same locally. If space is taken to be the nine-manifold $\mathbf{R}^9$ with coordinates $x^1, \ldots, x^9$, the D-branes can be a threebrane located at $x^4 = x^5 = \ldots = x^9 = 0$ and a fivebrane located at $x^1 = x^2 = x^3 = 0$, $x^4 = b$, where our present problem corresponds to $b \neq 0$, but in section 4.2 we will consider a case with $b = 0$. The ground state of a string connecting these branes is fermionic because (given the boundary conditions at the two ends) the ground state energy in the Neveu-Schwarz sector is positive while in the Ramond sector it is zero. The ground state is nondegenerate because there are no fermion zero modes in the Ramond sector.

replaces a small sphere around the origin in $\mathbf{R}^6$ by a copy of $\mathbf{RP}^5 = \mathbf{S}^5/\mathbf{Z}_2$.[5] So it replaces the $\mathbf{S}^5$ factor in the near-horizon geometry by $\mathbf{RP}^5$. So one is led (see [7, 8, 9]) to suspect that $\mathcal{N} = 4$ super Yang-Mills theory with orthogonal or symplectic gauge group can be described in terms of Type IIB superstring theory on an $AdS_5 \times \mathbf{RP}^5$ orientifold.

The statement that the $AdS_5 \times \mathbf{RP}^5$ model is an orientifold means that in going around a noncontractible loop in the target space, the orientation of the string world-sheet is reversed. A formal way to say this is the following. Let x be the generator of $H^1(\mathbf{RP}^5, \mathbf{Z}_2)$, which is isomorphic to $\mathbf{Z}_2$. Let Σ be a string world-sheet (a closed and not necessarily orientable two-dimensional surface), and let $w_1(\Sigma) \in H^1(\Sigma, \mathbf{Z}_2)$ be the obstruction to its orientability. Then we consider only maps $\Phi : \Sigma \to AdS_5 \times \mathbf{RP}^5$ such that $\Phi^*(x) = w_1(\Sigma)$.

Since the $\mathbf{S}_2$ action on $\mathbf{S}^5$ is free, the $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$ orientifold has no orientifold fixed points and thus no open string sector. There also is no "winding sector" consisting of closed strings wrapped around a non-contractible loop in $\mathbf{RP}^5$. (The latter statement is a general property of orientifolds, whether the $\mathbf{Z}_2$ action is free or not; we recall the reason at the beginning of section 3.3) The spectrum of the model, for weak coupling, is just the $\mathbf{Z}_2$-invariant part of the spectrum of the $AdS_5 \times \mathbf{S}^5$ compactification.

The interactions are, however, different, because in the orientifold the string world-sheet Σ need not be orientable. A basic case of an unorientable closed world-sheet is $\Sigma = \mathbf{RP}^2$, which we can identify as the quotient of the two-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ by the overall sign change $x_i \to -x_i$. A typical map $\Phi : \mathbf{RP}^2 \to \mathbf{RP}^5$ that obeys the orientifold constraint $\Phi^*(x) = w_1(\mathbf{RP}^2)$ is the embedding

$$(x_1, x_2, x_3) \to (x_1, x_2, x_3, 0, 0, 0)\,. \tag{3.1}$$

For large N, the leading deviation of the $AdS_5 \times \mathbf{RP}^5$ theory from the $\mathbf{Z}_2$-invariant part of the $AdS_5 \times \mathbf{S}^5$ theory comes from such $\mathbf{RP}^2$ worldsheets.[6]

The structure just described agrees beautifully with expectations from gauge theories. In the framework of 't Hooft [12], $SO(N)$ and $Sp(N/2)$ gauge theories should be described for large N by the "same" string theory that governs the $SU(N)$ theory, except that the strings, while still closed, are unoriented. This conclusion follows by analyzing, as in [21], the Riemann surfaces that are built from Feynman diagrams in the $SO(N)$ and $Sp(N/2)$ cases. Hence the spectra of the $SO(N)$ and $Sp(N/2)$ theories should be obtained from the spectrum of the $SU(N)$ theory by extracting the part invariant under an orientifold projection. The interactions of the $SO(N)$ and $Sp(N/2)$ gauge theories differ in perturbation theory from those of $SU(N)$, with the leading

[5] We recall that $\mathbf{RP}^n$ – real projective n-space – is the quotient of the sphere $x_1^2+x_2^2+\ldots+x_{n+1}^2 = 1$ by the $\mathbf{Z}_2$ symmetry $x_i \to -x_i$.

[6] The $\mathbf{RP}^2$ contribution dominates for large N over the one-loop contribution which obviously – since the $AdS_5 \times \mathbf{S}^5$ theory has additional intermediate states relative to $AdS_5 \times \mathbf{RP}^5$ – distinguishes the two theories. The $\mathbf{RP}^2$ amplitude is of order λ relative to the $\mathbf{S}^2$ contribution (λ being the string coupling constant), while the one-loop contribution is of relative order λ^2.

difference for large N coming from worldsheets (that is, Feynman diagrams) with $\mathbf{RP}^2$ topology. (See [21] for a detailed description of Feynman diagrams with $\mathbf{RP}^2$ topology.) Moreover, by analyzing the Feynman diagrams, one can show that the $SO(N)$ and $Sp(N/2)$ gauge theories differ from each other essentially just by the sign of the $\mathbf{RP}^2$ contribution.[7] This last statement is in accord with a feature of string theory on $AdS_5 \times \mathbf{RP}^5$ that we will see presently.

3.2. $SL(2,\mathbf{Z})$ Invariance and discrete torsion

The first important point about the Type IIB orientifold threeplane is that there exists a supersymmetric orientifold threeplane that is invariant under the $SL(2,\mathbf{Z})$ S-duality symmetry group of Type IIB superstrings. (This is implicit in the use [24] of this orientifold to explain Montonen-Olive duality for orthogonal and symplectic gauge groups.) To see this, recall first that the expectation values of the Type IIB dilaton and axion fields break $SL(2,\mathbf{Z})$ down to a finite subgroup that generically is generated by the element -1 of the center of $SL(2,\mathbf{Z})$. This group element, which we will call w, is of order two as an element of $SL(2,\mathbf{Z})$. But on spinors it generates a transformation of order four, not of order two. In fact, one has $w^2 = (-1)^F$, not $w^2 = 1$. This can be seen by inspection of the low energy Type IIB supergravity. If $Q_{\alpha\,L}$ and $Q_{\alpha\,R}$ are the left and right-moving supercharges of the theory (here $\alpha = 1, 2, \ldots, 16$ is a positive chirality spinor index of $SO(1,9)$; the spinor index will sometimes be suppressed), then w acts by $wQ_L = Q_R$, $wQ_R = -Q_L$. Or more succinctly, if we combine Q_L and Q_R to a doublet Q_i, $i = 1, 2$, we get $wQ_i = \epsilon_{ij}Q_j$; here $\epsilon_{ij} = -\epsilon_{ji}$ is the antisymmetric tensor with $\epsilon_{12} = 1$.

Now under an orientifolding operation that creates an orientifold threeplane, the supersymmetries transform by

$$M : Q_{\alpha i} \to (\Gamma^0\Gamma^1\Gamma^2\Gamma^3)_{\alpha\beta} m_{ij} Q_{\beta j}\,, \tag{3.2}$$

with some matrix m_{ij} that must map supersymmetries coming from left-movers to those coming from right-movers. So in the basis Q_L, Q_R, m looks like

$$m = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \tag{3.3}$$

for some a, b. The supersymmetry algebra obeyed by the Q's implies that a and b are each ± 1. For M to leave invariant some supersymmetry, we must pick m such that $M^2 = 1$; if instead $M^2 = -1$ all supersymmetry is broken. As $(\Gamma^0\Gamma^1\Gamma^2\Gamma^3)^2 = -1$, this forces $m^2 = -1$, so that $a = -b = \pm 1$. Depending on the sign, either m equals the $SL(2, Z)$-invariant matrix w described in the last paragraph, or $m = w^3$. (The

[7] In [22, 23], it is shown that Feynman diagrams of $Sp(N/2)$ are obtained from those of $SO(N)$ via the simple transformation $N \to -N$. Since a diagram describing a Riemann surface of genus g glued to s copies of $\mathbf{RP}^2$ is of order N^{2-2g-s}, the effect of $N \to -N$ is precisely to include a factor of -1 for each $\mathbf{RP}^2$.

same argument shows that an orientifold five-plane, for example, cannot be $SL(2,\mathbf{Z})$-invariant; in that case, one needs $m^2 = 1$, making it impossible for m to commute with w.)

By using this $SL(2,\mathbf{Z})$-invariant orientifolding operation in the presence of threebranes, we can get an $SL(2,\mathbf{Z})$-invariant configuration of threebranes on $\mathbf{R}^4 \times \mathbf{R}^6/\mathbf{Z}_2$, and hence (after taking the near-horizon geometry) an $SL(2,\mathbf{Z})$-invariant compactification on $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$. This is, however, not the only possible model on $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_2$. It is possible to make additional models by turning on "discrete torsion."

To understand the possibilities, we must first understand how the two two-form fields of the $SL(2,\mathbf{Z})$ theory – the Neveu-Schwarz B field B_{NS} and the Ramond-Ramond B field B_{RR} – transform under the orientifolding operation. First of all, because the orientifolding exchanges left- and right-movers on the string world-sheet, it reverses the world-sheet orientation. Hence B_{NS} does not transform as an ordinary two-form under orientifolding; the $\mathbf{Z}_2$ action multiplies B_{NS} by an extra minus sign. We can describe this by saying that B_{NS} is a twisted two-form: it is a section of $\Omega^2 \otimes \epsilon$, where Ω^2 is the bundle of ordinary two-forms, and ϵ will denote the unorientable real line bundle over $\mathbf{RP}^5$. Because the orientifolding is $SL(2,\mathbf{Z})$-invariant, B_{RR}, which is related to B_{NS} by $SL(2,\mathbf{Z})$, likewise is a twisted two-form.

In fact, since m acts on bosons as the element $-1 \in SL(2,\mathbf{Z})$,[8] it reverses the sign of all string and fivebrane charges. So both onebrane and fivebrane orientations are reversed in going around a non-trivial loop in $\mathbf{RP}^5$. This fact will play an important role in the present paper.

Topology of the B-field

Let us temporarily consider a Type IIB string theory that has *not* been orientifolded. In such a theory, the gauge-invariant field strength of a two-form field B is a three-form $H = dB$. Being closed, it determines a cohomology class $[H]$ that takes values in $H^3(M,\mathbf{R})$, where M is the spacetime manifold. When discrete torsion is taken into account, the cohomology class that measures the topology of the B-field is actually an element of $H^3(M,\mathbf{Z})$. We denote it in general as $[H]$ and call it the characteristic class of the B-field.

Now return to the case of an orientifold. In this case, B is a twisted two-form. The gauge-invariant field strength is still $H = dB$, but now H is a twisted three-form (a section of $\Omega^3 \otimes \epsilon$, with Ω^3 the bundle of ordinary three-forms). H is still closed; it determines a cohomology class $[H]$ that now takes values not in $H^3(M,\mathbf{R})$, but in $H^3(M,\tilde{\mathbf{R}})$, where $\tilde{\mathbf{R}}$ is the constant sheaf $\mathbf{R}$ twisted by ϵ (it is the sheaf of locally constant sections of ϵ). When discrete torsion is included, the topological type of the B-field is measured not by an element of $H^3(M,\mathbf{Z})$, as in the previous paragraph, but by an element $[H]$ of $H^3(M,\tilde{\mathbf{Z}})$, where $\tilde{\mathbf{Z}}$ is a twisted sheaf of integers. Like $\tilde{\mathbf{R}}$, $\tilde{\mathbf{Z}}$ is built using the same ± 1-valued transition functions used in defining the real line bundle

[8]Since $w^2 = (-1)^F$ acts trivially on bosons, w and w^3 both act on bosons as $-1 \in SL(2,\mathbf{Z})$.

ϵ; thus concretely, as one goes around a noncontractible loop in $\mathbf{RP}^5$, a section of $\tilde{\mathbf{Z}}$ (or $\tilde{\mathbf{R}}$) comes back to itself with a reversal of sign.

For many subsequent applications, we will need the homology and cohomology of $\mathbf{RP}^5$ with ordinary and twisted coefficients. A basic fact here is that an $\mathbf{RP}^i$ subspace of $\mathbf{RP}^5$, defined by a linear embedding $(x_1, x_2, \ldots, x_{i+1}) \to (x_1, x_2, \ldots, x_{i+1}, 0, \ldots, 0)$, is orientable or unorientable depending on whether i is odd or even. For odd i, the embedded $\mathbf{RP}^i$ determines an element of $H^i(\mathbf{RP}^5, \mathbf{Z})$, and for even i it defines an element of $H^i(\mathbf{RP}^5, \tilde{\mathbf{Z}})$. For $1 \leq i \leq 4$, these subspaces define two-torsion elements that generate the respective homology groups. The non-zero homology groups are thus

$$\begin{aligned} H_1(\mathbf{RP}^5, \mathbf{Z}) &= H_3(\mathbf{RP}^5, \mathbf{Z}) = \mathbf{Z}_2, \\ H_2(\mathbf{RP}^5, \tilde{\mathbf{Z}}) &= H_4(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2\,, \end{aligned} \tag{3.4}$$

along with $H_0(\mathbf{RP}^5, \mathbf{Z}) = H_5(\mathbf{RP}^5, \mathbf{Z}) = \mathbf{Z}$, $H_0(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$.

We will also have some use for the cohomology groups. As $\mathbf{RP}^5$ is orientable, Poincaré duality tells us that

$$\begin{aligned} H_i(\mathbf{RP}^5, \mathbf{Z}) &= H^{5-i}(\mathbf{RP}^5, \mathbf{Z}) \\ H_i(\mathbf{RP}^5, \tilde{\mathbf{Z}}) &= H^{5-i}(\mathbf{RP}^5, \tilde{\mathbf{Z}})\,. \end{aligned} \tag{3.5}$$

Hence we have, in particular,

$$H^3(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2\,. \tag{3.6}$$

The four models

We can now classify the possible models. The discrete torsion for either B field is classified by an element of $H^3(AdS_5 \times \mathbf{RP}^5, \tilde{\mathbf{Z}})$ which (because AdS_5 is contractible) is the same as $H^3(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$. Hence, for either of the two B fields, there is precisely one possible non-trivial choice of discrete torsion.

One can describe explicitly what the choice of discrete torsion means. In general, including a B field means that the path integral for a string world-sheet Σ (an elementary string or D-string depending on whether we are considering B_{NS} or B_{RR}) receives an extra factor

$$\exp\left(i\int_\Sigma B\right). \tag{3.7}$$

In our problem, since the characteristic class of the B field is a torsion element, it can be represented by a B field for which the curvature $H = dB$ is zero. This is the choice that we want to make in order to get an $AdS_5 \times \mathbf{S}^5$ compactification with spacetime supersymmetry. With this choice, the phase factor in (3.7) depends only on the homology class represented by Σ. Since the orientation of Σ is supposed to change sign in going around a noncontractible loop in $\mathbf{RP}^5$, the homology class of Σ is an element of $H_2(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$. The nontrivial element can be represented by $\Sigma = \mathbf{RP}^2$, with a map to $\mathbf{RP}^5$ of the topological type given by a linear embedding as in eqn. (3.1). A zero-curvature B-field that is trivial for such an $\mathbf{RP}^2$ would be completely trivial. If non-trivial for such an $\mathbf{RP}^2$, the B field must multiply the world-sheet path integral precisely by -1. (The reason for this is that the homology class of the embedded $\mathbf{RP}^2$ is a two-torsion element – generating $H_2(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$ – and so must be weighted by a factor of square $+1$.) This leads to a complete description of the role of the B-field, as follows. Any closed Riemann surface Σ is the connected sum of an oriented surface with s copies of $\mathbf{RP}^2$, for some s. If Σ is mapped to $AdS_5 \times \mathbf{RP}^5$ by a map Φ that obeys $\Phi^*(x) = w_1(\Sigma)$, it represents the same element of $H_2(\mathbf{RP}^5, \tilde{\mathbf{Z}})$ as s disjoint, linearly embedded $\mathbf{RP}^2$'s. So the discrete torsion multiplies the path integral for such a Σ by a factor of $(-1)^s$.

We can now see that there are four possible models, depending on two independent choices: the discrete torsion both for B_{NS} and for B_{RR} has two possible values, zero or non-zero. Let us call the two types of discrete torsion θ_{NS} and θ_{RR}. The action of $SL(2,\mathbf{Z})$ on the four models is easy to identify, given what we know about B_{NS} and B_{RR}. The two B-fields transform in the two-dimensional representation of $SL(2,\mathbf{Z})$, so θ_{NS} and θ_{RR} transform in that two-dimensional representation, reduced mod 2. So the model with $(\theta_{NS}, \theta_{RR}) = (0,0)$ is $SL(2,\mathbf{Z})$-invariant, and the other three models are permuted by $SL(2,\mathbf{Z})$ like the three non-zero elements of $\frac{1}{2}\Lambda/\Lambda$, with Λ a lattice acted on by $SL(2,\mathbf{Z})$ in the natural way.

Now let us determine the gauge groups of the four models. A priori, we might have $SO(N)$ or $Sp(N/2)$ for some N. The easiest model to identify is the $(0,0)$ model. Since the $AdS_5 \times \mathbf{S}^5$ compactification without discrete torsion is $SL(2,\mathbf{Z})$-invariant, the corresponding four-dimensional gauge theory has Montonen-Olive self-duality. The $\mathcal{N} = 4$ theory with $SO(2k)$ gauge group has Montonen-Olive self-duality, while the $SO(2k+1)$ and $Sp(k)$ theories are exchanged under duality. So the model without discrete torsion is an $SO(2k)$ gauge theory.

Now let us analyze the other models. Turning on $\theta_{NS} \neq 0$ multiplies the $\mathbf{RP}^2$ contribution to the worldsheet path integral of fundamental strings by a factor of -1. This has the effect of exchanging symplectic and orthogonal gauge groups. We already know that one $\theta_{NS} = 0$ model (the $(0,0)$ model) has orthogonal gauge group. So the models with symplectic gauge group are the models with $\theta_{NS} \neq 0$; they are in other words the $(\theta_{NS}, \theta_{RR}) = (1/2, 0)$ and $(1/2, 1/2)$ models. Likewise the models with orthogonal gauge groups are $(0,0)$ and $(0,1/2)$. We have already determined that the $(0,0)$ model has gauge group $SO(2k)$ for some integer k; the $(0,1/2)$ model, being related by $SL(2,\mathbf{Z})$ duality to the $(1/2,0)$ $Sp(k)$ model, has instead gauge group $SO(2k+1)$. The whole picture is portrayed in figure 3 on the next page. Notice that in this analysis we recover the correspondence between $Sp(k)$ and $SO(2k+1)$ models and non-zero points in $\frac{1}{2}\Lambda/\Lambda$ that was found in studies of their Coulomb branches after soft breaking to $\mathcal{N} = 2$ supersymmetry [25, 26, 27].

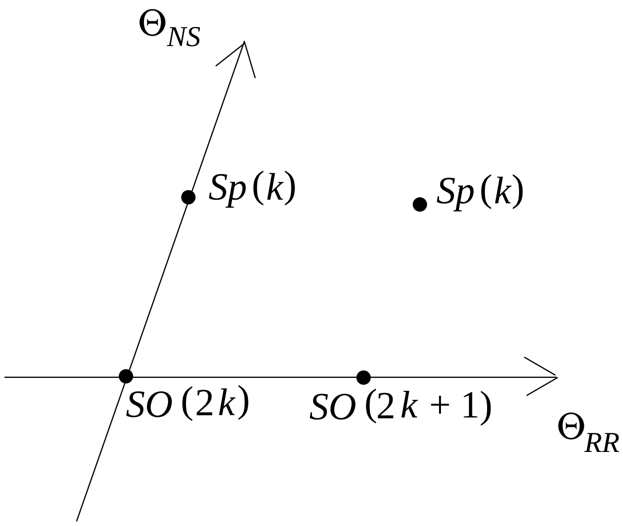

Figure 3: Sketched here are the four possible models, with zero and non-zero values of the discrete theta angles θ_{NS} and θ_{RR}. Each model is labeled by the corresponding gauge group. The coordinate axes have been slanted, because the four models correspond to the half-lattice points of an arbitrary rank two lattice in the complex plane.

3.3. Possibilities of brane wrapping

The rest of this paper will be concerned primarily with interpreting wrapped branes of various kinds in the $SO(N)$ and $Sp(N/2)$ gauge theories. First we make a crude classification that is independent of the discrete torsion, and then we consider an important refinement.

At first sight, it might appear that ten-dimensional strings (either elementary strings or D-strings) can be either wrapped on a one-cycle in $\mathbf{RP}^5$ or unwrapped, giving either a zero-brane or a one-brane on AdS_5. However, the wrapping modes that would give zero-branes are actually not allowed. Such a wrapping mode would correspond to a worldsheet of the form $S \times \mathbf{R}$, with S a circle mapped to a noncontractible loop in $\mathbf{RP}^5$ and $\mathbf{R}$ the time direction. As $S \times \mathbf{R}$ is orientable, such a world-sheet does not obey the condition that the orientation should be reversed in going around a noncontractible loop; and hence such wrapping modes are not present.

A more formal way to reach this conclusion is to state that since the string orientation is supposed to be reversed in going around the noncontractible loop, the strings are classified not by $H_1(\mathbf{RP}^5, \mathbf{Z})$ but by $H_1(\mathbf{RP}^5, \tilde{\mathbf{Z}})$. As this group vanishes, there is no nontrivial topological class of strings.

Now we consider the wrapping of threebranes. Since the threebrane charge is invariant under the orientifolding operation, the threebrane orientation is invariant in going around a loop in $\mathbf{RP}^5$. Hence wrapping modes of the threebrane are classified by the ordinary (untwisted) homology of $\mathbf{RP}^5$. To get from the threebrane an i-brane on AdS_5, we must wrap it on a $3-i$-cycle in $\mathbf{RP}^5$; these are classified by $H_{3-i}(\mathbf{RP}^5, \mathbf{Z})$. Looking back to (3.4), we see that there are three possibilities:

(i) A ten-dimensional threebrane that is not wrapped at all remains a threebrane in AdS_5.

(ii) The threebrane can be wrapped on a one-cycle, classified by $H_1(\mathbf{RP}^5, \mathbf{Z}) = \mathbf{Z}_2$, to give a two-brane on AdS_5.

(iii) The threebrane can be wrapped on a three-cycle, classified by $H_3(\mathbf{RP}^5, \mathbf{Z}) = \mathbf{Z}_2$, to give a particle on AdS_5.

Now we move on to fivebranes (for the moment NS and D fivebranes can be treated alike). Since fivebranes are dual to onebranes, the fivebrane charge is, like the onebrane charge, odd under going around a noncontractible loop in $\mathbf{RP}^5$. So wrapping modes of fivebranes are classified by twisted homology. We cannot consider a completely unwrapped fivebrane, since the six-dimensional world-volume of a fivebrane does not really fit into AdS_5 (which is only five-dimensional). So the possibilities are as follows:

(i)′ The fivebrane can be wrapped on a two-cycle, classified by $H_2(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$, to give a threebrane in AdS_5.

(ii)′ The fivebrane can be wrapped on a four-cycle, classified by $H_4(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$, to give a string in AdS_5.

The interpretation of the wrapping modes just described will be the main focus of the remainder of the present paper.

A topological restriction

The branes just described are subject to an important topological restriction. A Dirichlet fivebrane can be wrapped on an $\mathbf{RP}^4 \subset \mathbf{RP}^5$, to make a string, only if θ_{NS} vanishes; likewise, an NS fivebrane can be so wrapped only if θ_{RR} vanishes. And a threebrane can be wrapped on an $\mathbf{RP}^3 \subset \mathbf{RP}^5$, to make a particle, only if $\theta_{NS} = \theta_{RR} = 0$.

To explain these restrictions, we begin with the case of the NS fivebrane and the vanishing of θ_{NS}. We recall that the meaning of having $\theta_{NS} \neq 0$ is that, for a linearly embedded $\mathbf{RP}^2 \subset \mathbf{RP}^5$, one has

$$\exp\left(i\int_{\mathbf{RP}^2} B_{NS}\right) = -1\,. \tag{3.8}$$

What is written on the left hand side is in fact slightly oversimplified. The B_{NS} field is not really well-defined as a two-form; it is subject to the gauge transformations $B_{NS} \to B_{NS} + d\lambda$ with λ a twisted one-form. In general, B_{NS} can only be defined as a twisted two-form locally, after taking a cover of spacetime by small open sets; the twisted two-forms on different open sets are related by compatible gauge transformations in the overlaps of the open sets. The situation is somewhat analogous to that of a connection A on a $U(1)$ bundle; for any closed circle S in spacetime, one has a holonomy, usually written by physicists as

$$\exp\left(i\int_S A\right), \tag{3.9}$$

even though in fact A is only defined locally as a one-form.

In the presence of an NS fivebrane, the situation is changed as follows. There is a $U(1)$ gauge field a on the fivebrane world-volume. If f (defined locally as da) is the curvature of a, then the two-form $B' = B_{NS} - f$ is gauge-invariant,[9] and is hence well-defined as an ordinary, global, twisted two-form. B' is closed, since B_{NS} is closed and f is also closed by virtue of the Bianchi identity.

Now, suppose that the fivebrane is wrapped on an $\mathbf{RP}^4$, and deform the $\mathbf{RP}^2$ in (3.8) to be a subspace of this $\mathbf{RP}^4$. Because of the Dirac quantization condition on the flux of f, $\exp(i\int_{\mathbf{RP}^2} f) = 1$ (note that f is a twisted two-form and so can be integrated over the unorientable manifold $\mathbf{RP}^2$). So we have

$$\exp\left(i\int_{\mathbf{RP}^2} B_{NS}\right) = \exp\left(i\int_{\mathbf{RP}^2} B'\right). \tag{3.10}$$

However, for any closed globally defined twisted two-form B', one has

$$\int_{\mathbf{RP}^2} B' = 0. \tag{3.11}$$

In fact, this integral is determined by the cohomology class of B' in $H^2(\mathbf{RP}^4, \tilde{\mathbf{Z}})$, but this group vanishes.

So (3.10) implies that

$$\exp\left(i\int_{\mathbf{RP}^2} B_{NS}\right) = 1 \tag{3.12}$$

or in other words that the Dirichlet fivebrane can be wrapped on an $\mathbf{RP}^4 \subset \mathbf{RP}^5$, to make a string in AdS_5, only if θ_{NS} vanishes. Precisely the same argument (considering the path integral in a sector with a D-string worldvolume wrapped on an $\mathbf{RP}^2 \subset \mathbf{RP}^4$) shows that the NS fivebrane can be wrapped on such an $\mathbf{RP}^4$, to make a string, only if θ_{RR} vanishes. The same argument (considering wrapping on $\mathbf{RP}^2 \subset \mathbf{RP}^3$ of a fundamental or D-string worldvolume) shows that a threebrane can be wrapped on $\mathbf{RP}^3$ to make a particle on AdS_5 only if $\theta_{NS} = \theta_{RR} = 0$. Here one uses $H^2(\mathbf{RP}^3, \tilde{\mathbf{Z}}) = 0$ to show vanishing of (3.11).

There is, however, no such restriction on wrapping of fivebranes on $\mathbf{RP}^2 \subset \mathbf{RP}^5$ to make threebranes. Since $H^2(\mathbf{RP}^2, \tilde{\mathbf{Z}}) = \mathbf{Z}$, (3.11) need not vanish. There is no restriction on wrapping of a threebrane on $\mathbf{RP}^1$, since in this case $\mathbf{RP}^2$ cannot even be deformed into the threebrane.

General formulation and curvature correction

The above has been formulated in a rather *ad hoc* fashion. The basic idea is really the following. In the situation considered above, along the brane world-volume Y, we asserted the existence of a gauge-invariant field $B' = B - da$ that is gauge-equivalent to B. Existence of such a field means that the characteristic class $[H]$ of the B-field, when restricted to Y, vanishes as an element of $H^3(Y, \tilde{\mathbf{Z}})$. For instance, as a differential form the field strength of B is $H = dB'$, and this is certainly zero in cohomology. However, existence of a globally-defined B' field that is gauge-equivalent to B means that $[H]$ vanishes, when restricted to Y, as an integral class, not just in real cohomology. The

[9] Under $B_{NS} \to B_{NS} + d\lambda$, a transforms as $a \to a + \lambda$.

analog of this assertion for one-form fields is perhaps more familiar. If a complex line bundle $\mathcal{L}$ has a connection that can be represented by a globally-defined one-form A, then $\mathcal{L}$ is topologically trivial and $c_1(\mathcal{L})$ vanishes in integral cohomology, not just in real cohomology.

It was essential in deriving the restrictions on brane wrapping that the restriction of $[H]$ to Y vanishes in integral (and not just real) cohomology, since in the situation we considered, $[H]$ is anyway a torsion class. So let us look at this point closely. We write i for the inclusion of Y in spacetime, and $i^*([H])$ for the restriction of $[H]$ to Y. The gauge invariance of $B' = B - da$ with respect to local gauge transformations is certainly well understood in perturbative string theory. However, we should ask about invariance under global gauge transformations. Failure of such invariance would be a sort of "global anomaly" in the gauge-invariance of B', depending on the topology of Y, and might mean that $i^*([H])$ is not zero but is a torsion class determined by Y. ($i^*([H])$ would have to be a torsion class, since the formula $B' = B - da$ is certainly well enough understood to show vanishing of $i^*([H])$ at the level of differential forms.) If from the differential topology of Y, we can build in a natural way a torsion class $W \in H^3(Y, \tilde{\mathbf{Z}})$, then we should consider the possibility that the correct global restriction is not that $i^*([H]) = 0$ but rather that

$$i^*([H]) = W\,. \tag{3.13}$$

A possible W can be built as follows. Begin with the second Stieffel-Whitney class $w_2(Y) \in H^2(Y, \mathbf{Z}_2)$. Consider the exact sequence

$$0 \to \tilde{\mathbf{Z}} \xrightarrow{2} \tilde{\mathbf{Z}} \to \mathbf{Z}_2 \to 0\,, \tag{3.14}$$

where the first map is multiplication by 2 and the second is reduction modulo 2. The "connecting homomorphism" in the long exact sequence of cohomology groups that is derived from (3.14) maps w_2 to an element $W \in H^3(Y, \tilde{\mathbf{Z}})$. We can thus contemplate that the correct global condition is $i^*([H]) = W$ rather than $i^*([H]) = 0$. Whether such a correction is needed to cancel worldsheet global anomalies is under investigation [28]. Such a correction would be more or less analogous to the gravitational shift in the quantization law of the four-form field strength of M-theory [29]. We will describe in section 4 a further contribution to (3.13) that arises when additional branes of lower dimension end on the threebrane or fivebrane.

For most of this paper, it will not matter whether the correct condition on H contains the correction term W or not. The reason for this is that we will be considering branes wrapped on $\mathbf{RP}^i$ with $i \leq 4$ or $\mathbf{S}^5$, and these spaces all have $W = 0$. However, when we come to the question of stability of the baryon vertex for orthogonal and symplectic gauge groups, we will reach a situation in which the correction term is apparently needed to make AdS string theory match correctly with gauge theory predictions.

4. Gauge theory and branes on $\mathbf{RP}^5$

In this section, we will consider the various types of branes in $AdS_5\times\mathbf{RP}^5$, and interpret them in orthogonal and symplectic gauge theory. We consider in turn particles, strings, domain walls, the baryon vertex, and instantons.

4.1. The Pfaffian

First, we consider particles obtained by wrapping a threebrane on $\mathbf{RP}^3\subset\mathbf{RP}^5$. Since this $\mathbf{RP}^3$ represents a generator of $H^3(\mathbf{RP}^5,\mathbf{Z})=\mathbf{Z}_2$, the particle obtained this way is stable, but the number of such particles is conserved only modulo two; a pair of such particles can annihilate.

Now we apply the topological restriction found in section 3.3. A threebrane can wrap on $\mathbf{RP}^3$ only if both B-fields are topologically trivial, that is only if $\theta_{NS}=\theta_{RR}=0$. In view of the classification of the four models that was summarized in figure 3, this means that the gauge group is $SO(2k)$ for some k, and not $SO(2k+1)$ or $Sp(k)$. k is of course the number of five-form flux quanta on $\mathbf{RP}^5$:

$$\int_{\mathbf{RP}^5}\frac{G_5}{2\pi}=k\,. \tag{4.1}$$

On the double cover $\mathbf{S}^5$, the number of quanta is $N=2k$.

So we have found particles, conserved modulo two, that exist precisely for $SO(2k)$ gauge group, and not for the other symplectic or orthogonal groups. There are in gauge theory such states – the Pfaffians, described in the introduction (created by acting on the vacuum with a gauge-invariant operator of the form $\mathrm{Pf}(\Phi)$, with Φ a field in the adjoint representation and Pf the Pfaffian). One naturally suspects that the threebrane wrapped on $\mathbf{RP}^3$ should be identified with the Pfaffian particle of $SO(2k)$ gauge theory.

The number of Pfaffian objects is of course conserved modulo two for a simple group-theoretical reason. $\mathcal{N}=4$ super Yang-Mills theory with gauge group $SO(N)$ actually has $O(N)$ symmetry; the quotient $O(N)/SO(N)\cong\mathbf{Z}_2$ behaves as a global symmetry group. If τ is the generator of this $\mathbf{Z}_2$, then the Pfaffian is odd under τ and cannot decay to objects that are even under τ. Note that any gauge-invariant state constructed with less than $N/2$ elementary quanta is even. Since the global symmetry group is $\mathbf{Z}_2$, Pfaffians can annihilate in pairs, just like wrapped threebranes.

As further evidence for identifying Pfaffians with wrapped threebranes, we will determine the quantum numbers of the low-lying states on the two sides under the R-symmetry group of the theory. First we begin in the AdS description, where the R-symmetry group is a cover of the symmetry group of $\mathbf{RP}^5$. The manifold $\mathbf{RP}^5$ (identified as the sphere $\sum_{i=1}^6 x_i^2=1$ modulo the $\mathbf{Z}_2$ symmetry $x_i\to -x_i$) has a symmetry group $G_0=SO(6)/\mathbf{Z}_2$. A given $\mathbf{RP}^3$ subspace, say $x_5=x_6=0$, is invariant under $H_0=(SO(4)\times SO(2))/\mathbf{Z}_2$ (where $SO(4)$ acts on the first four coordinates and $SO(2)$ on the last two). The space of such embeddings is thus the homogeneous space G_0/H_0, which is the same as G/H, with $G=SO(6)$ and $H=SO(4)\times SO(2)$.

To find the low-lying threebrane quantum states, we must "quantize the collective coordinates," and analyze the quantum mechanics on G/H. The quantum wave states are not ordinary functions on G/H, but sections of a line bundle of degree k. The line bundle appears because the threebrane is electrically charged with respect to the five-form field strength G_5, and is of degree k (that is, it is the k^{th} power of the most basic line bundle) because the number of flux quanta on $\mathbf{RP}^5$ is k. The Hamiltonian acting on such functions is a G-invariant Laplacian. In our present case, G/H is a symmetric space, and there is only one invariant Laplacian.

We can identify G/H as the G manifold, with H acting on the right and G on the left. Functions on G/H are simply H-invariant functions on G, that is functions $\psi(g)$ with $\psi(g) = \psi(gh)$ for all $h \in H$. A section of a line bundle on G/H is a function on the G manifold that obeys

$$\psi(gh) = \psi(g)r(h)\,. \tag{4.2}$$

Here $h \to r(h)$ is a homomorphism of H to $U(1)$; the choice of homomorphism determines the line bundle. In our case, with $H = SO(4) \times SO(2)$, the most general homomorphism of H to $U(1)$ is the product of the "charge s" representation of $SO(2)$ (for some s) and the trivial representation of $SO(4)$. We want to set $s = k$, to get a line bundle of degree k, so for us ψ should be $SO(4)$-invariant and should transform with charge k under $SO(2)$.

If we identify $G = SO(6)$ as the group of 6×6 orthogonal matrices $g^i{}_j$, $i, j = 1, \ldots, 6$, then functions on the G manifold can be expanded as polynomials in the matrix elements of G. The matrix elements $g^i{}_j$ themselves transform as $(\mathbf{6}, \mathbf{6})$ under $SO(6) \times SO(6)$, and as $(\mathbf{6}, \mathbf{4})^{\mathbf{0}} \oplus (\mathbf{6}, \mathbf{1})^{\mathbf{1}} \oplus (\mathbf{6}, \mathbf{1})^{-\mathbf{1}}$ under $SO(6) \times SO(4) \times SO(2)$; here the exponent is the $SO(2)$ charge. To make sections of the desired line bundle, we want polynomials in the $g^i{}_j$ of $SO(2)$ charge k. To minimize the energy, that is the eigenvalue of the Laplacian, we must select the polynomials of lowest degree that have charge k. These are simply the polynomials of degree k in the $(\mathbf{6}, \mathbf{1})^{\mathbf{1}}$. Note that these polynomials transform in the *traceless* symmetric product of k copies of the $\mathbf{6}$; the trace terms vanish because g is an orthogonal matrix.

The manifold $SO(6)/(SO(4) \times SO(2))$ is actually a homogeneous Kahler manifold, and the line bundle just considered is ample.[10] Hence the lowest-lying wavefunctions just found are holomorphic sections of the line bundle. They consequently give BPS states. It is therefore possible to compare them to chiral operators in the boundary conformal field theory. Such operators can be constructed as in [30] from the scalar fields of the $\mathcal{N} = 4$ super Yang-Mills theory on the boundary. The scalars transform in the $\mathbf{6}$ of the $SO(6)$ global symmetry and in the adjoint of the gauge group $SO(2k)$. The Pfaffian of the scalars (that is, $\epsilon^{a_1 a_2 \ldots a_{2k}} \Phi^{i_1}_{a_1 a_2} \Phi^{i_2}_{a_3 a_4} \ldots \Phi^{i_k}_{a_{2k-1} a_{2k}}$, where the i's are

[10]To be more precise, $SO(6)/(SO(4) \times SO(2))$ is the same as $SU(4)/(SU(2) \times SU(2) \times U(1))$ and is the Grassmannian of complex two-planes in $\mathbf{C}^4$; it can also be described as a quadric in $\mathbf{CP}^5$. The line bundle described in the last paragraph is the usual very ample line bundle over this Grassmannian or quadric.

$SO(6)$ indices and the a's are gauge indices) transforms in the k^{th} symmetric product of the **6**. The traceless part of the k^{th} symmetric product can be shown to consist of BPS operators by considering an $\mathcal{N} = 1$ subalgebra of the supersymmetry algebra.

Because of the BPS property, it is not really necessary to compare the precise masses and dimensions of operators, but it is instructive to work out the order of magnitude. A threebrane wrapped on a volume V has a mass m in AdS units of order V/λ, with λ the string coupling constant. According to [2, 3] the dimension of the corresponding conformal field theory operator is of order mR, with R the radius of curvature of AdS. In the present case, $V \sim R^3$ and $R \sim (\lambda k)^{1/4}$, so the dimension is k in order of magnitude, and independent of λ. This agrees with conformal field theory, where the Pfaffian operator has dimension precisely k. The BPS property of course ensures that the coefficient of k also works out correctly.

String ending on threebrane

So far we have considered wrapped threebranes without strings. But a basic property of the threebrane is that a string can end on it. Let us consider a situation in which a single elementary Type IIB string, as in figure 4, ends on a threebrane that is wrapped on $\mathbf{RP}^3$. We suppose that the other end of the string is connected to an elementary quark (in the fundamental representation of the gauge group) on the boundary of AdS space.

Consider, in general, a Type IIB string ending on a D-brane. Generally, the end of the string is electrically charged with respect to the $U(1)$ gauge field a living on the D-brane. Let f be the field strength of a. In the special case of a threebrane, we can make a duality transformation to a dual $U(1)$ gauge field $\tilde{a}$ and field strength $\tilde{f} = *f$.

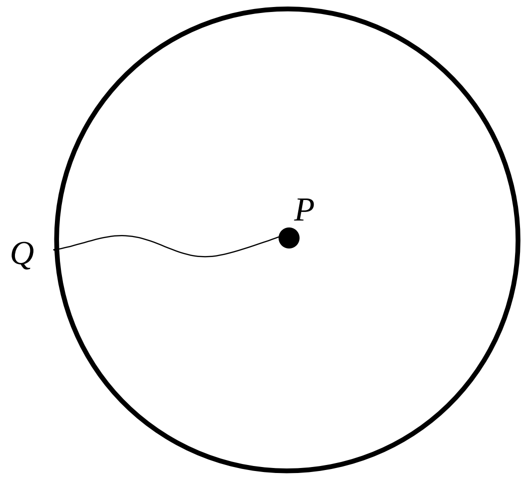

Figure 4: For gauge group $SO(2k + 1)$, a threebrane wrapped on $\mathbf{RP}^3$ must have an odd number of strings ending on it. Sketched in the figure is a threebrane, at a point in AdS space labeled P, connected to the boundary by an elementary string that terminates on a boundary point labeled Q.

The endpoint of the Type IIB string is magnetically charged with respect to $\tilde{f}$:

$$d\tilde{f} = 2\pi\delta_P\,, \tag{4.3}$$

where δ_P is a delta function supported at the endpoint of the string. (δ_P is normalized so that its integral is 1; its coefficient is 2π, since the string carries one flux quantum.)

We recall now that the field strength H_{RR} of the Ramond-Ramond two-form B_{RR} is in the absence of any branes $H_{RR} = dB_{RR}$. Along the brane, there is a gauge-invariant version $B'_{RR} = B_{RR} - \tilde{f}$ of the B-field. Rewriting the definition of H_{RR} in terms of B'_{RR}, we get

$$H_{RR} = dB'_{RR} - 2\pi\delta_P\,. \tag{4.4}$$

There is no such magnetic correction for B_{NS}, whose field strength remains

$$H_{NS} = dB'_{NS}\,. \tag{4.5}$$

A point $P \in \mathbf{RP}^3$ generates $H_0(\mathbf{RP}^3, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$, and its Poincaré dual δ_P generates $H^3(\mathbf{RP}^3, \tilde{\mathbf{Z}})$, which is likewise $\mathbf{Z}_2$. So the formulas (4.4), (4.5) say that in the presence of the string, H_{RR} is topologically non-trivial while H_{NS} is trivial; thus the wrapped threebrane, with a single string ending on it, is possible only if $\theta_{RR} \neq 0$, $\theta_{NS} = 0$. This configuration is in other words possible if and only if the gauge group is $SO(2k+1)$ for some k.[11]

This has a natural interpretation in gauge theory. From a field Φ in the adjoint representation of $SO(2k+1)$, we cannot make a gauge-invariant Pfaffian. But we can couple k copies of Φ to make a field $\epsilon^{ab_1b_2\dots b_{2k}}\Phi_{b_1b_2}\dots\Phi_{b_{2k-1}b_{2k}}$ in the fundamental representation of the gauge group. So in an $SO(2k+1)$ gauge theory (such as the $\mathcal{N}=4$ supersymmetric theory) in which all elementary fields are in the adjoint representation, an external quark in the fundamental representation can be screened, by a combination of k gluons. In the AdS description, this screening is described by connecting the external quark to a wrapped threebrane by an elementary Type IIB string.

A more sophisticated way to describe (4.4) is that the trivialization of the B-field along the threebrane fails in the presence of a string boundary. If C is the curve on the threebrane worldvolume along which the string ends (thus $C = P \times \mathbf{R}$ in a static situation, with $\mathbf{R}$ parametrized by time), then the condition $i^*([H]) = 0$ discussed at the end of section 3 is corrected in the presence of the string to

$$i^*([H_{RR}]) = [C]\,, \tag{4.6}$$

with $[C]$ the Poincaré dual to C. In this equation, C is the boundary of the elementary strings that end on the threebrane; there is a similar equation with H_{RR} replaced by H_{NS} and C replaced by the boundary of the D-strings ending on the threebrane.

[11] Note that a B-field on $\mathbf{RP}^3$ whose field strength H has the delta function contribution in (4.4) (and which is trivial away from P) is topologically equivalent to a B-field whose field strength vanishes and which assigns the value -1 to a wrapped $\mathbf{RP}^2$. They are topologically equivalent as there is in fact only one nonzero element of $H^3(\mathbf{RP}^3, \tilde{\mathbf{Z}})$.

Particles and operators

A clarification should be added here. Let us go back to the $SO(2k)$ case, and reconsider the threebrane wrapped on $\mathbf{RP}^3$ with no strings attached. Such a wrapped threebrane is a particle from the AdS point of view, but does not quite have that interpretation in the boundary theory because of the absence of a mass gap. Describe AdS_5 by the metric $ds^2 = f(x_0)(dx_0^2 + \sum_{i=1}^4 dx_i^2)$, where $f(x_0) = 1/x_0^2$, and the x_i, $i > 0$ should be understood as the coordinates of the boundary theory which is defined at $x_0 = 0$. An AdS particle of mass m, located at a given x_0, has energy $m\sqrt{f(x_0)}$ from the point of view of the boundary theory; because $f(x_0) \to 0$ for $x_0 \to \infty$, this can be arbitrarily small. Hence the wrapped threebranes considered above should be associated in the boundary conformal field theory with Pfaffian operators, which acting on the vacuum can create states of arbitrarily small energy. Suppose that as in various constructions in [16], one replaces AdS_5 by a similar metric with $f(x_0)$ bounded strictly above zero, and let f_0 be the minimum value of f. Then the boundary theory has a mass gap, and a particle of mass m in the interior theory gives rise to a particle of mass roughly mf_0 in the boundary theory. By this mechanism, in a suitable context, one would describe actual Pfaffian particles in gauge theory on the boundary in terms of wrapped branes in the interior.

4.2. Strings

We now consider strings in AdS_5 that arise by wrapping a Dirichlet fivebrane on $\mathbf{RP}^4 \subset \mathbf{RP}^5$. We will call these strings fat strings to avoid confusing them with elementary strings and D-strings.

According to the criterion in section 3.3, fivebrane wrapping on $\mathbf{RP}^4$ is only possible if $\theta_{NS} = 0$, that is if the gauge group is orthogonal rather than symplectic. Fat strings can annihilate in pairs, because they are classified by $H_4(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$. Their tension is proportional to the Dirichlet fivebrane tension, and so is of order $1/\lambda$, that is, of order N.[12]

Strings with precisely these properties are expected in the boundary conformal field theory. They are strings associated with external charges in the spinor representation of the gauge group. This representation does not arise in the tensor product of any number of copies of the N-dimensional or fundamental representation of $SO(N)$, so an external spinor charge is associated with a new kind of string that cannot decay to the strings associated with charges in the fundamental representation. The new strings are conserved only modulo two, since the tensor product of two spinors can be

[12] As in the last paragraph of section 4.1, this string tension in AdS space will become a string tension in the boundary theory only if the boundary theory is perturbed to have $f_0 > 0$. But even without making such a perturbation, the factor of N is observable in the boundary theory. For example, in the boundary conformal field theory, the energy of a state with external spinor charges at specified locations on the boundary is of order N, since it receives a contribution from the fat strings connecting them in the interior of AdS space.

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decomposed as a sum of tensor products of the fundamental representation. No such strings are expected for symplectic gauge group, since the symplectic group has no "new" representations beyond what one finds in the tensor products of the fundamental representation. It is natural for the string associated with an external spinor charge to have tension of order N, since the highest weight vector of the spinor representation (which is the vector $(1/2, 1/2, \ldots, 1/2)$, with $N/2$ entries) has length of order $\sqrt{N}$. Indeed, in a current algebra description [31], strings associated with external spinor charges are seen as open string solitons (which can sometimes be deformed to D-branes) and have tensions of order N.

All of these facts encourage the idea that the fat string of AdS_5 is related to spinor charges on the boundary. For more such evidence, we will now go back to the orientifold whose near-horizon geometry is $AdS_5 \times \mathbf{RP}^5$. By studying the orientifold, we will also get some clues that will enable us to more precisely match group theory with the wrapped fivebrane.

Orientifold and fivebrane

We consider the familiar $\mathbf{R}^4 \times (\mathbf{R}^6/\mathbf{Z}_2)$ orientifold. In $\mathbf{R}^4$ we take coordinates $x^0, \ldots, x^3$, and in $\mathbf{R}^6$ we take coordinates $x^4, \ldots, x^9$. The $\mathbf{Z}_2$ acts by $x^i \to -x^i$, $i = 4, \ldots, 9$. There are N threebranes at $x^4 = \ldots = x^9 = 0$.

The $\mathbf{R}^6/\mathbf{Z}_2$ factor is interpreted as follows in $AdS_5 \times \mathbf{RP}^5$. The radial function $\rho = \sqrt{\sum_{i=4}^{9} x_i^2}$ of $\mathbf{R}^6/\mathbf{Z}_2$ becomes one of the AdS_5 coordinates, the other four being $x^0, \ldots, x^3$. The angular directions in $\mathbf{R}^6/\mathbf{Z}_2$ are identified with $\mathbf{RP}^5$.

Now consider a fivebrane whose world-volume is specified by $x^1 = x^2 = x^3 = x^9 = 0$, with arbitrary values of x^0 and of $x^4, \ldots, x^8$. From the $AdS_5 \times \mathbf{RP}^5$ point of view, such a fivebrane is wrapped on an $\mathbf{RP}^4 \subset \mathbf{RP}^5$ and looks like a fat string on AdS_5. The $\mathbf{RP}^4$ in question is the subspace of $\mathbf{RP}^5$ with $x^9 = 0$. The fat string worldsheet in AdS_5 is parametrized by ρ and x^0 and is at $x^1 = x^2 = x^3 = 0$.

The $5-3$ strings connecting the fivebrane to the threebrane are, in their ground state, fermions [20]; we already exploited this fact in section 2 in our discussion of the baryon vertex. In the present case, since the fivebrane and the threebranes actually meet at $x^1 = \ldots = x^9 = 0$, the ground state of the $5-3$ string has zero energy. Because of the orientifolding, the $5-3$ and $3-5$ strings are actually equivalent. The ground states of these strings give, overall, N fermion zero modes in the fundamental representation of $SO(N)$. Upon quantizing these fermion zero modes, we learn that the ground state of the system transforms in the spinor representation of $SO(N)$ (as we will discuss momentarily, both chiralities appear if N is even). We denote the fermion zero modes as $\psi_1, \ldots, \psi_N$; they generate a Clifford algebra.

The fivebrane, interpreted as a string in AdS_5, has an endpoint at $\rho = 0$. This endpoint lies on the boundary of AdS_5, so this is an example of a fat string that ends at a boundary point of AdS_5. We have seen, in this particular example, that there is a charge in the spinor representation of $SO(N)$ at the boundary point. This gives strong confirmation for the proposal that in general strings of this type can terminate at points on the boundary of AdS_5 at which there are external spinor charges.

In the absence of orbifolding, there is a $U(1)$ gauge field on the fivebrane worldvolume; the $5-3$ and $3-5$ strings have respectively charge 1 and -1. Orbifolding reverses the sign of the $U(1)$, exchanging the $5-3$ and $3-5$ strings. The gauge symmetry on the fivebrane world-volume is broken down to $\mathbf{Z}_2$. The $5-3$ strings are all odd under this $\mathbf{Z}_2$, and in particular that is true for the Clifford algebra generators $\psi_1, \ldots, \psi_N$. The symmetry of the Clifford algebra under which the generators are all odd is, for even N, usually called chirality – it assigns the value $+1$ or -1 to spinors that transform in the two different spin representations of $SO(N)$. For odd N, there is only one spin representation, but there are two inequivalent representations of the Clifford algebra – distinguished by the sign of the product $\psi_1\psi_2\ldots\psi_N$, which commutes with the Clifford algebra – and the operation $\psi_i \to -\psi_i$ exchanges these two representations. The physical interpretation of the appearance of both representations of the Clifford algebra is somewhat unclear, so in returning to AdS_5, I will consider only the case of even N.

Return to AdS_5

We consider in AdS_5 a string, made by wrapping a fivebrane on $\mathbf{RP}^4$, that connects two boundary points of AdS_5. There is an external spinor charge at each end of the string. Since the unbroken $\mathbf{Z}_2$ along the string is a gauge symmetry, the overall quantum state of the string should be $\mathbf{Z}_2$-invariant, provided we include the charges at the ends of the string. We take this to mean that the product of the total $\mathbf{Z}_2$ charge of the string with the chiralities of the spinor charges at the ends of the string equals $+1$.

Hence if, as in figure 5, a fat string absorbs an elementary string, its chirality is reversed. In fact, the endpoint of an elementary string on a fivebrane has charge ± 1 under the $U(1)$ gauge field on the fivebrane – and so is odd under the unbroken $\mathbf{Z}_2$. If, in joining on an elementary string to the fivebrane, the $\mathbf{Z}_2$ quantum number to the left

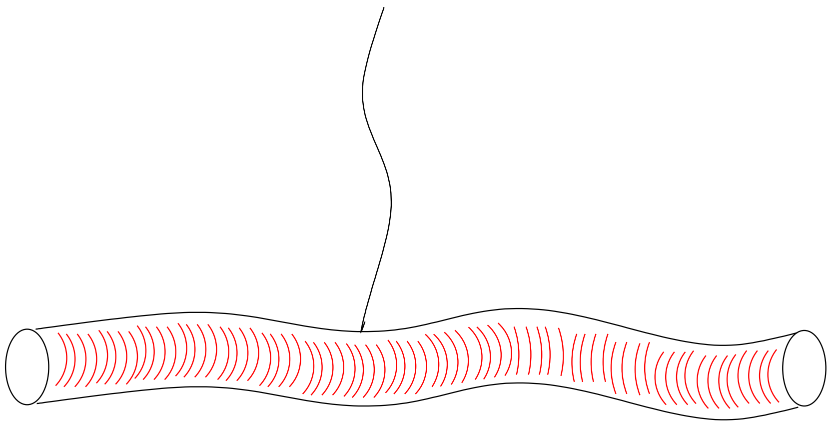

Figure 5: A “fat string” in AdS space with a thin or elementary string ending on it. The fat string reverses chirality when absorbing an elementary string.

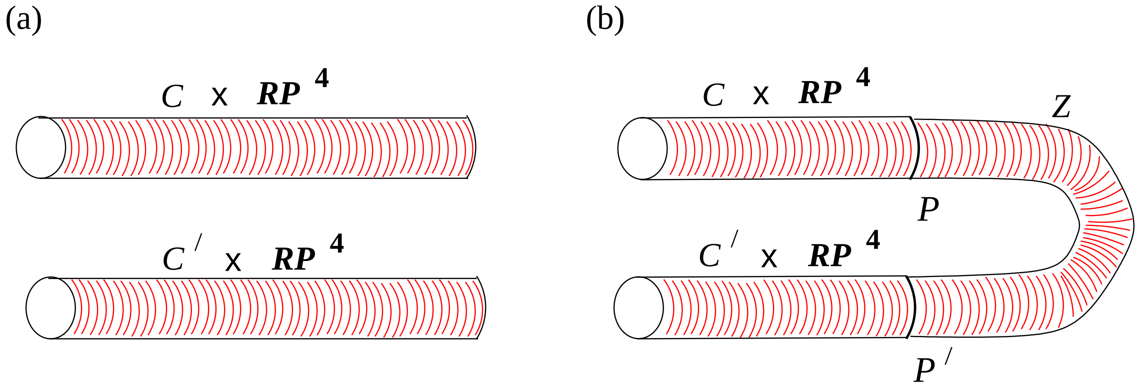

Figure 6: Sketched in (a) are two roughly parallel fat strings in AdS space. Sketched in (b) is a process, described in the text, in which the fat strings annihilate.

of the junction is kept fixed, then the chirality of the spinor charge on the right must be reversed to preserve overall $\mathbf{Z}_2$ neutrality.

This is in agreement with the following group-theoretical fact. Let S_+ and S_- denote the positive and negative chirality representations of $SO(N)$ (for even N), and let V be the vector representation. Then $V \otimes S_+$ contains S_-, but not S_+, and conversely $V \otimes S_-$ contains S_+. In other words, the chirality of a spinor is reversed whenever a vector is absorbed, just as we find in analyzing the fat strings.

We now wish to reproduce one additional fact of group theory: the tensor product $S_+ \otimes S_+$ is a sum of representations of the form $\wedge^s V$ (the s^{th} antisymmetric tensor power of V) with s congruent to $N/2$ mod 2. In other words, when two identical fat strings annihilate, the number of elementary strings produced should be $N/2$ mod 2. We will see this in the following way.

As a preliminary, recall that a fivebrane wrapped on $\mathbf{RP}^4$ is unorientable. For fivebranes in $AdS_5 \times \mathbf{RP}^5$, what must be oriented is not the tangent bundle TX of the fivebrane worldvolume X, but $TX \otimes \epsilon$, where ϵ is the pullback to X of the unorientable real line bundle over $\mathbf{RP}^5$. For any manifold Z with tangent bundle TZ, we call an orientation of $TZ \otimes \epsilon$ a "twisted orientation" of Z.

Now, consider as in figure 6(a), two identical adjacent fivebranes, whose worldvolumes, at given time, are of the form $C \times \mathbf{RP}^4$ and $C' \times \mathbf{RP}^4$. Here C and C' are two roughly parallel paths in AdS_5, and for each fivebrane we use the same $\mathbf{RP}^4$.[13] For example, if $\mathbf{RP}^5$ is obtained by projectivizing a copy of $\mathbf{R}^6$ with coordinates $x_1, \ldots, x_6$, then $\mathbf{RP}^4$ can be defined as the subspace

$$x_6 = 0\,. \tag{4.7}$$

[13]Instead of thinking of the configuration in figure 6(a) as a time-independent configuration with strings running from left to right, one might alternatively want to think of the AdS_5 coordinate that runs from left to right in this picture as the "time" coordinate, while an additional "space" coordinate that will be common to all fivebranes is suppressed in the figure. This way of looking at the figure agrees better with the terminology we will use about "annihilation" of fat strings.

To ensure that the fivebranes are identical, we want to give the "same" twisted orientation to $C \times \mathbf{RP}^4$ and $C' \times \mathbf{RP}^4$, using the fact that C and C' are parallel and the two $\mathbf{RP}^4$'s are identical.

To describe the annihilation of the two fivebranes, we wish to suppose that as in figure 6(b), C and C' are semi-infinite paths, terminating at points P and P', respectively. We pick a five-manifold Z in spacetime whose boundary is the union of $P \times \mathbf{RP}^4$ and $P' \times \mathbf{RP}^4$. Z should have a twisted orientation which agrees on the boundary with those of $C \times \mathbf{RP}^4$ and $C' \times \mathbf{RP}^4$. Then figure 6(b) describes the termination of the two semi-infinite fat branes upon arriving in the vicinity of Z. We call this annihilation of the fat strings.

What can we pick for Z? One is tempted to try $Z = D \times \mathbf{RP}^4$ where D is a path in AdS_5 from P to P', parametrized say by an angle θ, with $0 \leq \theta \leq \pi$, and $\mathbf{RP}^4$ is still the subspace (4.7) of $\mathbf{RP}^5$. With this choice of Z, however, the twisted orientations in figure 6(b) would not match. The problem arises because as both C and C' are "incoming," their annihilation via D involves a reversal of orientation. The $\mathbf{RP}^4$ factor, being constant, has inevitably a constant twisted orientation, so consideration of this factor does not help.

To fix things, one must let the $\mathbf{RP}^4$ factor vary in such a way that as one goes from $\theta = 0$ to $\theta = \pi$, $\mathbf{RP}^4$ comes back with the opposite twisted orientation, thus accounting for the orientation reversal that is involved in letting C and C' annihilate. To make this happen, we replace (4.7) with the θ-dependent condition

$$\cos\theta\, x_6 + \sin\theta\, x_5 = 0\,. \tag{4.8}$$

This describes for every θ an $\mathbf{RP}^4$, which coincides at $\theta = 0$ or $\theta = \pi$ with the original $\mathbf{RP}^4$ defined in (4.7). But starting with a given twisted orientation at $\theta = 0$, one comes back at $\theta = \pi$ with the opposite twisted orientation. One way to show this is that, as $\mathbf{RP}^5$ is orientable, to give a twisted orientation to $\mathbf{RP}^4$ is the same as giving a twisted orientation of its normal bundle. This is concretely a one-form, nonzero in the direction normal to $\mathbf{RP}^4$, that is odd under sign change of all x_i. The one-form $\cos\theta\, dx_6 + \sin\theta\, dx_5$ does the job. It obviously is continuous in θ and odd under sign reversal of the x_i, and has opposite sign at $\theta = \pi$ relative to $\theta = 0$, confirming that the twisted orientation is reversed in going around this path.

Now we want to count the elementary strings produced in the annihilation of the two incoming fat strings. The reason that such elementary strings are created is somewhat similar to the reason that the wrapped fivebrane studied in section 2 behaves as a baryon vertex. The key ingredient in section 2 was that fiveform flux integrated over the fivebrane contributes to the charge that couples to the $U(1)$ field on the fivebrane. In the present context, this means that the fiveform flux integrated over Z equals the total violation of chirality in the annihilation of the two fat strings, and hence the number of elementary strings produced, modulo two. We recall that from group theory, this number should be $N/2$, modulo two.

To compute the total charge violation, we note that the map from Z to $\mathbf{RP}^5$ is generically one-to-one. (This is so because for a generic value of $(x_1, x_2, \ldots, x_6) \in \mathbf{RP}^5$, the equation $\cos\theta\, x_6 + \sin\theta\, x_5 = 0$ is obeyed for a unique value of θ with $0 \leq \theta \leq \pi$.) So the flux integral over Z equals that on $\mathbf{RP}^5$. On $\mathbf{RP}^5$ there are $N/2$ units of five-form flux (N on the covering space $\mathbf{S}^5$). So chirality is violated by $N/2$ units, modulo two, in agreement with expectations from gauge theory.[14]

BPS property

As a prelude to discussing the BPS properties of elementary and fat strings, let us consider ten-dimensional supersymmetric Yang-Mills theory. The supersymmetry transformation law for the gauge field A is

$$\delta A_i = \epsilon^\alpha \Gamma_{i\,\alpha\beta} \lambda^\beta\,, \tag{4.9}$$

with λ the gluino, ϵ an anticommuting parameter, Γ a gamma matrix, and α, β spinor indices of $SO(1,9)$. This transformation law shows that if n is a null vector, then $n \cdot A$ is invariant under eight supersymmetries, namely those associated with parameters ϵ such that $\epsilon^\alpha n^i \Gamma_{i\,\alpha\beta} = 0$. It follows that if C is a lightlike straight line in $\mathbf{R}^{10}$, then

$$\mathrm{Tr}_R P \exp \int_C A \tag{4.10}$$

is invariant under eight supersymmetries, for any representation R of the gauge group.

Ten-dimensional super Yang-Mills can be dimensionally reduced to four-dimensional super Yang-Mills with $\mathcal{N} = 4$ supersymmetry. Four components of the ten-dimensional gauge field reduce to the four-dimensional gauge field, which we will still call A, and the other six components become scalars ϕ_i, $i = 1, \ldots, 6$ in the adjoint representation. Let now D be a spacelike straight line in $\mathbf{R}^4$, and let $\vec{m}$ be a unit six-vector, given in components by m_i, $i = 1, \ldots, 6$ with $\sum_i m_i^2 = 1$. A four-dimensional analog of the statement that (4.10) is invariant under eight supersymmetries is the statement that

$$\mathrm{Tr}_R P \exp \int_D (A + i\vec{m} \cdot \vec{\phi}) \tag{4.11}$$

is invariant under eight supersymmetries (plus, in fact, eight superconformal symmetries). In essence, (4.11) is the dimensional reduction of (4.10), using a complex null vector whose components are the unit tangent vector to D together with $i\vec{m}$.

To make contact with an AdS description, it is convenient to work with Euclidean signature and to add to $\mathbf{R}^4$ a point at infinity, making $\mathbf{S}^4$ – the boundary of AdS space. Including the point at infinity, D becomes a great circle on $\mathbf{S}^4$. What BPS configuration in AdS space corresponds to a Wilson line that wraps around D? D is

[14] A more rigorous version of this discussion could be given using mod two cohomology instead of differential forms and would count the chirality violation modulo two.

the boundary of an AdS_2 subspace of AdS_5.[15] To make a BPS state invariant under the symmetries that preserve the AdS_2, we need a brane on $AdS_5 \times \mathbf{S}^5$ or $AdS_5 \times \mathbf{RP}^5$ whose worldvolume will be this AdS_2 times a suitable submanifold of $\mathbf{S}^5$ or $\mathbf{RP}^5$.

If R is the fundamental representation of $SU(N)$, $SO(N)$, or $Sp(N)$, then in the proposal of [1, 5], one simply uses an elementary Type IIB string, placed at the point $\vec{m}$ of $\mathbf{S}^5$ or $\mathbf{RP}^5$. The worldsheet of the BPS configuration is $AdS_2 \times \{\vec{m}\}$. What do we do if R is the spinor representation of $SO(N)$? In this case, we must consider a fat string, whose worldvolume will be $AdS_2 \times \mathbf{RP}^4$, with some $\mathbf{RP}^4$ subspace of $\mathbf{RP}^5$. To make it possible to interpret the BPS operator (4.11) via AdS fat strings, the choice of a unit vector $\vec{m}$ must determine a particular $\mathbf{RP}^4 \subset \mathbf{RP}^5$. Happily, it does: the $\mathbf{RP}^4$ in question is simply the subspace of $\mathbf{RP}^5$ that is "orthogonal" to $\vec{m}$. (In other words, this $\mathbf{RP}^4$ is parametrized by x_i, $i = 1, \ldots, 6$, defined up to overall sign, with $\sum_i x_i^2 = 1$ and $\sum_i m_i x_i = 0$.) This relies on the fact that we built the fat string by wrapping precisely on a codimension one subspace of $\mathbf{RP}^5$; the codimension one property is a non-trivial check, since it was determined on grounds (namely, the dimensions of fivebranes and of $\mathbf{RP}^5$) that are seemingly unrelated to the BPS properties of loop operators.

4.3. Domain walls

In this subsection, we consider objects in $AdS_5 \times \mathbf{S}^5$ and $AdS_5 \times \mathbf{RP}^5$ that look like threebranes in the five noncompact dimensions of AdS_5. In $AdS_5 \times \mathbf{S}^5$, the only such object is the unwrapped Type IIB threebrane. In $AdS_5 \times \mathbf{RP}^5$, in addition to the unwrapped Type IIB threebrane, we have threebranes made by wrapping a Type IIB fivebrane on $\mathbf{RP}^2 \subset \mathbf{RP}^5$.

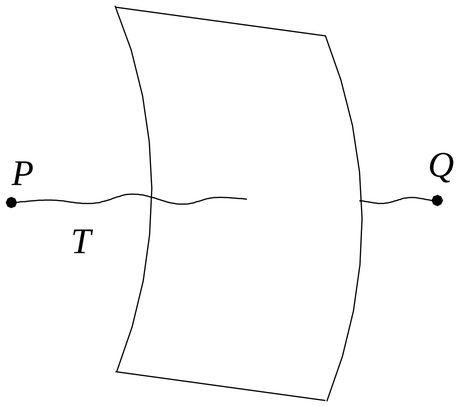

Figure 7: A path T connecting two points P and Q that are on opposite sides of a domain wall.

AdS_5 has four spatial dimensions, so a threebrane has codimension one and could potentially behave as a domain wall, with the string theory vacuum "jumping" as one crosses the threebrane. We will see that all threebranes mentioned in the last paragraph are domain walls in that sense.

We begin by considering $AdS_5 \times \mathbf{S}^5$. The only threebrane is the unwrapped Type IIB threebrane. It is a source of the five-form field G_5 of Type IIB string theory. From this, it can be deduced that in crossing the threebrane, the flux of G_5 over $\mathbf{S}^5$ jumps by one unit. The argument runs as follows. Let P and Q be points on opposite sides of the threebrane, as in figure 7. Let T be a path from P to Q, intersecting the threebrane

[15]In $\mathbf{R}^5$ with coordinates $y_1, \ldots, y_5$ and $|\vec{y}| = \sqrt{\sum_i y_i^2}$, one can regard $\mathbf{S}^4$ as the space $|\vec{y}| = 1$, and AdS_5 as the space $|\vec{y}| < 1$ with metric $4d\vec{y}^2/(1-|\vec{y}|^2)^2$. One can take for D the great circle in $\mathbf{S}^4$ given by the equations $y_3 = y_4 = y_5 = 0$, and for AdS_2 the subspace of AdS_5 defined by the same equations.

once. The six-manifold $T \times \mathbf{S}^5$ intersects the threebrane at a single point. We consider the integral

$$\int_{T\times\mathbf{S}^5} \frac{dG_5}{2\pi} \,. \tag{4.12}$$

This integral equals 1, since $dG_5/2\pi$ is a delta function supported on the threebrane, and as we have noted, the threebrane intersects $T\times\mathbf{S}^5$ at one point. On the other hand, we can evaluate the integral using Stokes's theorem. Since the boundary of $T \times \mathbf{S}^5$ is $P \times \mathbf{S}^5 - Q \times \mathbf{S}^5$ (where the minus sign keeps track of the relative orientation), we get

$$\int_{P\times\mathbf{S}^5} \frac{G_5}{2\pi} - \int_{Q\times\mathbf{S}^5} \frac{G_5}{2\pi} = 1 \,. \tag{4.13}$$

This establishes the claim that the integrated five-form flux jumps by one unit when one crosses the threebrane.

Thus the vacuum is different on the two sides. The difference is easy to describe intuitively: the gauge group of the boundary conformal field theory is determined by the five-form flux, so it is $SU(N)$ on one side, and $SU(N\pm 1)$ on the other side.

Domain walls in $AdS_5 \times \mathbf{RP}^5$

Now we consider the somewhat more subtle case that $\mathbf{S}^5$ is replaced by $\mathbf{RP}^5$. First we consider the unwrapped threebrane. This can be treated just as before. The flux integral over $\mathbf{RP}^5$

$$\int_{\mathbf{RP}^5} \frac{G_5}{2\pi} \tag{4.14}$$

changes by 1 in crossing the threebrane. So on the double cover $\mathbf{S}^5$ of $\mathbf{RP}^5$, the flux integral changes by two. Hence the gauge group jumps in crossing the threebrane from $SO(N)$ to $SO(N\pm 2)$, or from $Sp(k)$ (with $k = N/2$) to $Sp(k\pm 1)$.

Formally similar, but more subtle because torsion is involved, is a Dirichlet or Neveu-Schwarz fivebrane wrapped on $\mathbf{RP}^2 \subset \mathbf{RP}^5$ to make a threebrane. In this case, let X be the four-manifold $X = T\times\mathbf{RP}^3$, with T the same curve as before, and $\mathbf{RP}^3$ a generic $\mathbf{RP}^3 \subset \mathbf{RP}^5$. Since a generic $\mathbf{RP}^2$ and $\mathbf{RP}^3$ in $\mathbf{RP}^5$ have one point of intersection, X generically intersects the fivebrane at one point. The boundary of X is the union of the three-manifolds $P\times\mathbf{RP}^3$ and $Q\times\mathbf{RP}^3$. Because the fivebrane is a magnetic source for the B-field and intersects X in one point, the total "magnetic charge" of the B-field on the boundary of X is non-zero. If the B-field is trivial topologically on $P \times \mathbf{RP}^3$, it is nontrivial on $Q \times \mathbf{RP}^3$, and vice-versa. Thus, the discrete "theta angle" jumps in crossing the threebrane.

Which θ angle jumps depends on which fivebrane one considers. By wrapping a Dirichlet fivebrane on $\mathbf{RP}^2$, we get a domain wall across which θ_{RR} jumps; by wrapping an NS fivebrane on $\mathbf{RP}^2$, we get a domain wall across which θ_{NS} jumps. Most surprising, from a conventional field theory point of view, is the domain wall with a jump in θ_{NS}; the boundary conformal field theory has orthogonal gauge group on one side of this domain wall, and symplectic gauge group on the other.

An interesting property of domain walls made from fivebranes is that they can carry threebrane charge. We recall that on the fivebrane worldvolume X, there is a $U(1)$ gauge field a; its field strength is a twisted two-form f. The topology of the $U(1)$ gauge field is determined by a characteristic class $[f] \in H^2(X, \tilde{\mathbf{Z}})$. As is usual in brane theory, the fivebrane carries threebrane charge proportional to $[f]$. Because $H^2(\mathbf{RP}^2, \tilde{\mathbf{Z}}) = \mathbf{Z}$, a fivebrane wrapped on $\mathbf{RP}^2$ can carry arbitrary threebrane charge; that is, it can absorb any number of unwrapped threebranes. This gives domain walls across which the gauge group jumps from $Sp(k)$ to $Sp(k')$ with arbitrary k and k', or similarly from $SO(2k)$ to $SO(2k'+1)$ or $Sp(k')$.

Comparison to flat space orientifold

Like the fat strings of section 4.2, the domain walls made by wrapping fivebranes on $\mathbf{RP}^2$ can be conveniently studied by going back to the flat-space orientifold whose near-horizon geometry is $AdS_5 \times \mathbf{RP}^5$.[16] By doing so, we can help clarify an interesting phenomenon found [32] in applications of orientifolds to gauge theory. Hence in this discussion, we will consider orientifold k-planes, not just three-planes.

Consider in $\mathbf{R}^{10}$, with coordinates $x^0, x^1, \ldots, x^9$, a $\mathbf{Z}_2$ transformation that leaves fixed $x^0, \ldots, x^k$ and reverses the sign of the others. The fixed point set is called an orientifold k-plane (in Type IIA or Type IIB string theory for even or odd k). The space normal to the k-plane looks like $\mathbf{R}^{9-k}/\mathbf{Z}_2$. If we want to analyze the behavior of string theory in this space using only conventional geometry, we should keep away from the singularity at the origin. The exterior to the singularity is contractible to $\mathbf{RP}^{8-k}$. The exterior is in fact $\mathbf{R}^+ \times \mathbf{RP}^{8-k}$, where $\mathbf{R}^+$ is parametrized by the distance from the origin.

Since $H^3(\mathbf{RP}^{8-k}, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$, there is a possibility of a discrete theta angle in the exterior space. As we know from our discussion in section 3, turning on θ_{NS} has the effect precisely of reversing the sign of the elementary string amplitude for worldsheets of topology $\mathbf{RP}^2$ (and θ_{RR} is similarly related to D-strings).

In perturbative string theory, there are two types of orientifold k-planes. They differ by the sign of the orientifold projection for open strings – the two choices lead to orthogonal and symplectic gauge groups – and by the sign of the $\mathbf{RP}^2$ path integral. Because of the last assertion (in relation to the statement in the previous paragraph), the two types of k-plane differ by the value of θ_{NS} in the smooth manifold exterior to the k-plane. Via this interpretation, the two types of perturbative orientifold plane can be distinguished just by measurements outside the plane – though to observe and distinguish the gauge group directly takes measurements near the orientifold singularity. (In the case $k = 3$, if one considers also D-strings, one can make a further refinement and distinguish orientifold planes by the value of θ_{RR} in the exterior region.)

[16] As far as I know, the other main examples in the present paper, which are the baryon vertex and the Pfaffian particle, cannot be studied in a similar way. The stability of those objects depends on gravitational corrections to the flat-space orientifold.

Now consider an NS fivebrane whose world-volume is parametrized by $x^0, \ldots, x^{k-1}$ and by $x^{k+1}, \ldots, x^6$. The intersection of the fivebrane with the orientifold plane is of codimension one in that plane and divides it, potentially, into domain walls. It was in fact shown in [32] that in crossing such a fivebrane, the "type" of the orientifold plane is reversed. In our present language, this means that the value of θ_{NS} jumps in crossing the fivebrane. The explanation is just as above (indeed, our previous discussion corresponds to the case $k = 3$, while $k = 4$ was considered in [32]). In the directions normal to the orientifold plane, the fivebrane is wrapped on $\mathbf{R}^+ \times \mathbf{RP}^{5-k}$. The Poincaré dual of $\mathbf{RP}^{5-k}$ in $\mathbf{RP}^{8-k}$ is the generator of $H^3(\mathbf{RP}^{8-k}, \tilde{\mathbf{Z}})$. The magnetic coupling of the B-field to the fivebrane means that in crossing the fivebrane, the characteristic class of the B-field changes by this class, or in other words θ_{NS} jumps.

4.4. The baryon vertex

Here we will study the baryon vertex of orthogonal and symplectic gauge theory – adapting to $\mathbf{RP}^5$ the considerations of section 2 in the $\mathbf{S}^5$ case.

By analogy with section 2, one expects at first sight that the baryon vertex will consist of a fivebrane wrapped once on $\mathbf{RP}^5$ – say a Dirichlet fivebrane if one wishes a baryon vertex connecting external quarks of the electric gauge group. Thinking along those lines, one quickly comes to a paradox. Suppose, for example, that we are doing $SO(2k)$ gauge theory. Then there are k units of five-form flux on $\mathbf{RP}^5$ ($2k$ units on the covering space $\mathbf{S}^5$). Assuming that the fivebrane wraps once on $\mathbf{RP}^5$ and following the reasoning of section 2, we then find that k elementary strings terminate on the fivebrane, and that those strings behave as fermions. We thus seem to obtain a "baryon vertex" with an antisymmetric coupling of k external quarks. But there is no gauge-invariant antisymmetric combination, in $SO(2k)$ gauge theory, of k external quarks. The baryon vertex of $SO(2k)$ gauge theory should couple $2k$ external quarks, not k of them.

What saves the day is that a state consisting of a fivebrane wrapped once on $\mathbf{RP}^5$ does not exist. Let x be the generator of $H^1(\mathbf{RP}^5, \mathbf{Z}_2)$ and X the fivebrane world-volume. Let also Φ be the map of X to $AdS_5 \times \mathbf{RP}^5$ given by the embedding[17] of the fivebrane in space-time. As we discussed in section 3.3, we are limited to X and Φ such that $\Phi^*(x) = w_1(X)$.

For most of the present section, we can ignore the AdS_5 factor, because it is contractible, and consider Φ as a map to $\mathbf{RP}^5$. For a static fivebrane, one has $X = Y \times \mathbf{R}$, with $\mathbf{R}$ the "time" direction and Y a five-manifold. This is contractible to Y, so topologically we can think of Φ as a map of Y to $\mathbf{RP}^5$. For a static fivebrane wrapped

[17] Actually, fivebranes of the topologies considered in the present section cannot be embedded in $AdS_5 \times \mathbf{RP}^5$. But they can be mapped to $AdS_5 \times \mathbf{RP}^5$ via maps that are embeddings except in codimension four. (One does this by letting the brane "wiggle" generically in AdS_5 while wrapping on $\mathbf{RP}^5$.) At points where the fivebrane is not an embedded submanifold, there will be low energy modes that cannot be seen using a long-wavelength fivebrane effective action; but the codimension of the singularities is too high for such phenomena to be relevant for us.

once on $\mathbf{RP}^5$, we have $Y = \mathbf{RP}^5$ and Φ the identity map. In this case, the condition $\Phi^*(x) = w_1(Y)$ is not obeyed, since, as $\mathbf{RP}^5$ is orientable, $w_1(\mathbf{RP}^5) = 0$, while for Φ the identity map, $\Phi^*(x) = x \neq 0$. This at least shows that we cannot get a contradiction by taking the fivebrane worldvolume to be $\mathbf{RP}^5$.

To show more generally that, regardless of the fivebrane topology, there is no baryon vertex coupling sk quarks for any odd integer s, we want to show that if Y is any closed five-manifold, and $\Phi : Y \to \mathbf{RP}^5$ any map that obeys $w_1(Y) = \Phi^*(x)$, then $\Phi(Y)$ wraps an even number of times around $\mathbf{RP}^5$. This follows from the fact that for any closed five-manifold Y, $w_1(Y)^5 = 0$.[18] Hence if $w_1(Y) = \Phi^*(x)$, one has $0 = w_1(Y)^5 = (\Phi^*(x))^5 = \Phi^*(x^5)$. But x^5 is the mod two fundamental class of $\mathbf{RP}^5$, and maps $\Phi : Y \to \mathbf{RP}^5$ with $\Phi^*(x^5) = 0$ are precisely those of even degree.

The basic non-trivial case is that Y wraps twice around $\mathbf{RP}^5$. As a simple example, we take $Y = \mathbf{S}^5$, with the natural two-to-one projection to $\mathbf{RP}^5$.[19] Since the degree of the map is two, the five-form field integrates over Y to twice the value on $\mathbf{RP}^5$. Hence the problem discussed above is avoided. As desired, Y gives an antisymmetric coupling of N external quarks, not $N/2$ of them, for $SO(N)$ or $Sp(N/2)$ gauge theory.

At this stage, however, we meet the following perplexing question. Y vanishes as an element of $H_5(\mathbf{RP}^5, \tilde{\mathbf{Z}})$, since that group is in fact zero. So why is the baryon vertex just found stable? Before trying to discuss this question in the case of AdS string theory, we will first review the situation in field theory.

Existence and stability of baryon vertex in field theory

In $SO(N)$ or $Sp(N/2)$ gauge theory, there is a fundamental representation of dimension N. The N-fold completely antisymmetric tensor product of this representation is gauge-invariant. If ψ is a fermion valued in the fundamental representation, this antisymmetric invariant is

$$B = \frac{1}{N!} \epsilon_{i_1 i_2 \dots i_N} \psi^{i_1} \psi^{i_2} \dots \psi^{i_N} . \tag{4.15}$$

This is the "baryonic" combination of N external quarks, which we have aimed to reproduce in AdS space via the "baryon vertex," at which N elementary strings can join. Note that the invariant B can be defined whether the gauge group is $SO(2k)$, $SO(2k+1)$, or $Sp(k)$. So a baryon vertex should exist for each of the possible groups.

This is in agreement with what we have found above. The manifold $Y = \mathbf{S}^5$ has $H^3(Y, \tilde{\mathbf{Z}}) = 0$, so the requirement that the B-fields should be topologically trivial when

[18] A proof using the Adem relations for Steenrod squares was provided by D. Freed. Using the fact that $Sq^k(w) = w^2$ for w a k-dimensional class, and that Sq^1, as a map to the top dimension, is the cup product with w_1, one has $w_1(Y)^5 = Sq^1 w_1(Y)^4 = Sq^1 Sq^2 w_1(Y)^2 = Sq^1 Sq^2 Sq^1 w_1(Y) = Sq^3 Sq^1 w_1(Y)$. In the last step, one of the Adem relations was used. But $Sq^3 Sq^1 w_1(Y) = 0$, since Sq^r annihilates an s-dimensional class for $r > s$.

[19] In other words, we regard $\mathbf{S}^5$ as the sphere $\sum_{i=1}^6 x_i^2 = 1$, and $\mathbf{RP}^5$ as the quotient of $\mathbf{S}^5$ by $x_i \to -x_i$. The "identity" map on the x_i gives the degree two map of $\mathbf{S}^5$ to $\mathbf{RP}^5$.

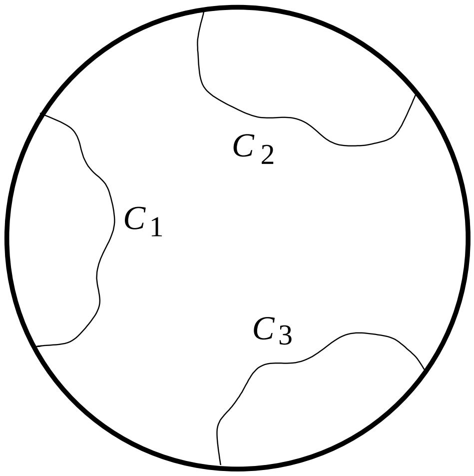

Figure 8: In $Sp(k)$ gauge theory – as sketched here for $k = 3$ – a baryon vertex can decay to a configuration in which $2k$ charges on the boundary are connected pairwise by elementary strings $C_1, \ldots, C_k$.

pulled back to Y is automatically obeyed. The use of Y for a baryon vertex is equally valid for any value of θ_{NS} or θ_{RR}. The AdS baryon vertex thus exists regardless of the gauge group of the boundary theory.

What about stability of the baryon in field theory? This is more complicated. In $Sp(N/2)$ gauge theory, there is an invariant second rank antisymmetric tensor γ_{ij}, via which one can form the "meson" $M = \frac{1}{2}\gamma_{ij}\psi^i\psi^j$. A "baryon" of $Sp(N/2)$ can decay to mesons since in fact

$$B = \frac{1}{(N/2)!} M^{N/2} \,. \tag{4.16}$$

Thus, we should expect no topological stability for the AdS baryon vertex when $\theta_{NS} \neq 0$. For $\theta_{NS} \neq 0$, an initial state with a fivebrane wrapped twice on $\mathbf{RP}^5$ and connected by N elementary strings to charges on the boundary should, topologically, be able to decay to a state (indicated in figure 8) with no fivebrane and with $N/2$ strings that join the external quarks pairwise. (This decay is not necessarily favored energetically.)

The case of $SO(N)$ is more subtle. The analog of γ_{ij} is the "metric," the symmetric tensor δ_{ij}. We must take account of the fact – already used in our discussion of Pfaffians in section 4.1 – that $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SO(N)$ actually has $O(N)$ symmetry, not just $SO(N)$. The generator τ of the quotient $O(N)/SO(N)$ behaves as a global symmetry. δ_{ij} is invariant under $O(N)$, while $\epsilon_{i_1 i_2 \ldots i_N}$ is odd under τ. So the baryon – which is odd under τ – cannot decay to mesons – which are even under τ. Hence, the transition sketched in figure 8 should be impossible for $SO(N)$, that is, for $\theta_{NS} = 0$.

On the other hand, in $SO(N)$ gauge theory there are states odd under τ other than the baryon. For N even, one has the "Pfaffian" combination of $N/2$ gauge bosons, which we interpreted in section 4.1 in terms of a wrapped three-brane. For N odd,

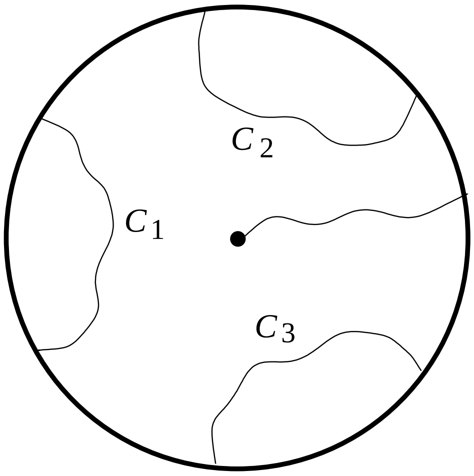

Figure 9: In $SO(2k)$ gauge theory, the baryon vertex can decay to a state containing a wrapped threebrane plus strings making pairwise connections between external charges. In $SO(2k+1)$, which is the case sketched here for $k = 3$, the final state contains in addition an odd number of strings connecting the wrapped threebrane – indicated as a heavy dot in the interior – to the boundary. In the figure, there is one such string.

the wrapped three-brane joined by a string to an external charge similarly represents a τ-odd state.

For $SO(N)$, decay to elementary strings only should be forbidden, but there should be processes in which a state containing a baryon vertex constructed from a wrapped fivebrane is transformed to a state containing a wrapped threebrane together with elementary strings, as sketched in figure 9. Both are odd under τ, so transitions between them should be possible. After analyzing, albeit in an incomplete fashion, the decay of the baryon vertex to elementary strings, we will briefly discuss the decay to a state containing a wrapped threebrane.

Decay of baryon vertex in AdS

To interpret the decay of a baryon to mesons in AdS space, we proceed as follows. We consider a state that up to time t_0 has a wrapped fivebrane with topology $\mathbf{S}^5$. Thus, this part of the fivebrane world-volume has topology $\mathbf{R}_- \times \mathbf{S}^5$, where $\mathbf{R}_-$ is the part of the "time" line with $t \leq t_0$. To describe the decay of the fivebrane, we find a compact six-manifold X, with boundary $\mathbf{S}^5$, and we take the total fivebrane worldvolume Y to be the manifold obtained by gluing $\mathbf{R}_- \times \mathbf{S}^5$ onto X along their common boundary. The map from Y to spacetime is specified by picking a map $\Phi : X \to AdS_5 \times \mathbf{RP}^5$ that maps the boundary to $\{t_0\} \times \mathbf{S}^5$. Thus, the fivebrane worldvolume comes in from the far past, and "ends" within a finite time of t_0. (What the attached strings are doing meanwhile will be discussed later.)

The map Φ, which topologically can be viewed as a map of X to $\mathbf{RP}^5$, must obey the usual condition $\Phi^*(x) = w_1(X)$, and must agree on the boundary of X with the usual two-to-one projection from $\mathbf{S}^5$ to $\mathbf{RP}^5$. (The last condition ensures that the maps from $\mathbf{R}_- \times \mathbf{S}^5$ and from X to spacetime glue correctly to a map from Y.) $\mathbf{S}^5$ with the two-to-one map to $\mathbf{RP}^5$ vanishes as an element of $H_5(\mathbf{RP}^5, \tilde{\mathbf{Z}})$ (as that group is actually zero), so there exists a manifold X and map Φ obeying the given conditions. An explicit example is as follows. In $\mathbf{R}^7$, with coordinates $x_1, \ldots, x_7$, take the subspace $\mathbf{S}^5 \times I$ (I is the unit interval) defined by $\sum_{i=1}^6 x_i^2 = 1$, $|x_7| \leq 1$. Divide $\mathbf{S}^5 \times I$ by the $\mathbf{Z}_2$ transformation $x_i \to -x_i$, $i = 1, \ldots, 7$; let X be the quotient. By forgetting x_7, X maps to $\mathbf{RP}^5$; this is the desired map Φ. The boundary of X is the double cover of $\mathbf{RP}^5$ given by taking $x_7 = \pm 1$. This double cover is $\mathbf{S}^5$ (since one can divide $X = \mathbf{S}^5 \times \{x_7 = \pm 1\}$ by $\mathbf{Z}_2$ by restricting x_7 to be $+1$), and Φ induces on $\mathbf{S}^5$ the usual two-to-one projection to $\mathbf{RP}^5$. This shows that X and Φ have the desired properties.

So we have found a mechanism for the decay of the fivebrane. However, we must impose the usual condition on topological triviality of the NS B-field when pulled back to a Dirichlet fivebrane worldvolume. For the baryon vertex itself, this was no problem, as $H^3(\mathbf{S}^5, \tilde{\mathbf{Z}}) = 0$. However, $H^3(X, \tilde{\mathbf{Z}})$ is non-trivial, so there is a potential obstruction to decay of the baryon vertex via the manifold X. In fact, X is contractible to $\mathbf{RP}^5$ (by squeezing the x_7 axis down to zero), so $H^3(X, \tilde{\mathbf{Z}})$ is naturally isomorphic to $H^3(\mathbf{RP}^5, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$. The map $\Phi : X \to \mathbf{RP}^5$ is actually a homotopy equivalence, and Φ^* is therefore an isomorphism.

If $[H]$ is the characteristic class of the NS B-field, then the condition stated in section 3.3, namely $i^*([H]) = 0$, implies that $[H] = 0$. This means that the decay of the baryon vertex by this mechanism is possible if and only $\theta_{NS} = 0$. The decay would, in other words, occur for orthogonal and not for symplectic gauge groups, which is precisely the wrong answer! This suggests that we should look for an "overall minus sign" that will exchange the two cases. It was, in fact, suggested at the end of section 3.3 that the general condition is really $i^*([H]) = W$, with W a certain natural element of $H^3(X, \tilde{\mathbf{Z}})$. It can be shown[20] that for the manifold X, W is the generator of $H^3(X, \tilde{\mathbf{Z}})$, while $W = 0$ for all other brane worldvolumes considered in the present paper. Thus, if the proper condition is $i^*([H]) = W$, then brane decay by the mechanism discussed here is actually possible if and only if $\theta_{NS} \neq 0$, that is, if and only if the gauge group is symplectic. This is a strong hint that the proper condition involves the W term, a matter that is under investigation [28].

It remains to discuss what happens to the strings while the fivebrane is being capped off by the manifold X. The baryon vertex on $\mathbf{R}_- \times \mathbf{S}^5$ is connected to the boundary of AdS_5 by N strings whose worldsheets end in curves C_i on $\mathbf{R}_- \times \mathbf{S}^5$. One can take

[20] The normal bundle to $\mathbf{RP}^5 \subset X$ is the unorientable real line bundle over $\mathbf{RP}^5$; also, X is homotopic to $\mathbf{RP}^5$, so one can evaluate W by restricting to X. The total Stieffel-Whitney class of X, taking account the normal bundle to $\mathbf{RP}^5$, is $(1+x)^7$, where x is the generator of $H^2(\mathbf{RP}^5, \mathbf{Z}) = \mathbf{Z}_2$. In particular, $w_2(X) = x$ and is non-zero. Now consider the long exact cohomology sequence derived from (3.14). $w_2(X)$ cannot be lifted to a class in $H^2(\mathbf{RP}^5, \tilde{\mathbf{Z}})$ (as that group vanishes), so $W \neq 0$ and hence generates $H^3(X, \tilde{\mathbf{Z}}) = \mathbf{Z}_2$. For all other cases in the present paper, $W = 0$ since either $w_2(X) = 0$ (branes wrapped on $\mathbf{RP}^3$ or $\mathbf{RP}^4$) or $H^3(X, \tilde{\mathbf{Z}}) = 0$ (branes wrapped on $\mathbf{RP}^2$).

these curves to be of the form $\mathbf{R}_- \times P_i$, with P_i some points in $\mathbf{S}^5$. To complete the description of the decay of the baryon vertex, the union of the C_i must be extended to a collection of curves without boundary on Y. This must be done by taking the N boundary points of the C_i on $\mathbf{S}^5$ (these are the points $t_0 \times P_i$), and connecting them pairwise via strings in X. If N is odd, this is impossible, as an odd number of points cannot be joined pairwise. This corresponds to the statement that in $SO(N)$ gauge theory with odd N, a "baryon" cannot decay to mesons simply because it contains an odd number of quarks; any decay of the baryon vertex will, as we have seen, involve a wrapped threebrane in the final state. For even N, however, there is no obstruction to joining the ends of the C_i pairwise (because X is unorientable, this can be done in a way that is compatible with the twisted orientations of the C_i; one merely loops around an orientation-reversing loop in X whenever needed). Once the C_i have been extended over Y, the resulting curves can be connected to the boundary of AdS space via string worldsheets, completing the description of the decay of the baryon vertex.

Decay to state containing wrapped threebrane

Now we will briefly analyze the decay of the baryon vertex to a state containing a threebrane wrapped on $\mathbf{RP}^3$. We recall from our group theory discussion that such a decay should be possible when the gauge group is $SO(N)$ – which is the case that the baryon vertex cannot decay to a state containing mesons only.

The basic reason that this is possible is that threebranes can end on fivebranes. Moreover, the end of the threebrane on a fivebrane worldvolume is a magnetic source for the $U(1)$ gauge field a that propagates on the fivebrane. Let X be the fivebrane worldvolume, E the worldvolume of a threebrane whose boundary is on X, and D the boundary of E. Then D is an orientable three-manifold; it is orientable because the threebrane worldvolume is always orientable, and the boundary of an orientable manifold is orientable. Consequently, the Poincaré dual of D is a class $[D] \in H^3(X, \tilde{\mathbf{Z}})$. Because the equation[21] $i^*([H]) = W$ is really the Bianchi identity for a, and D acts as a magnetic source for a, the equation becomes in the presence of a threebrane

$$i^*([H]) = W + [D]\,. \tag{4.17}$$

The $[D]$ term here is just analogous to the $[C]$ term in (4.6), which governs strings ending on threebranes. To be more precise, (4.17) holds for Dirichlet fivebranes with $[H] = [H_{NS}]$, which is the case we will actually consider, or for NS fivebranes with $[H] = [H_{RR}]$.

Now recall that the mechanism that prevents decay of the $SO(N)$ baryon vertex to strings only is that when the gauge group is $SO(N)$, one has $i^*([H]) = 0$; but W is not zero for the fivebrane worldvolume that describes decay of the baryon vertex. What happens if threebranes are included? We see from (4.17) that decay of the baryon

[21] In this discussion we will have to assume again that the W term is really present!

vertex to a state containing threebranes as well as strings is possible if $W+[D]=0$ or equivalently (since W is a two-torsion class) $[D]=W$.

This condition can be obeyed and corresponds, as expected, to having in the final state a single threebrane (or an odd number of them) wrapped on $\mathbf{RP}^3 \subset \mathbf{RP}^5$. Indeed, as the fivebrane worldvolume X is contractible to $\mathbf{RP}^5$, we can take it to contain a copy of $\mathbf{RP}^5$, say at some time t_1 and at some point P in the spatial part of AdS_5. Let D be any $\mathbf{RP}^3$ subspace of this $\mathbf{RP}^5$. Then $[D]=W$ (each of them being the nonzero element of $H^3(\mathbf{RP}^5, \tilde{\mathbf{Z}})$). To give a threebrane worldvolume E with boundary D, we simply let $\mathbf{R}_+$ be the product of the set in AdS_5 with $t \geq t_1$ and position P in space, and we take $E = \mathbf{R}_+ \times \mathbf{RP}^3$. Clearly, with this choice of E, we have decay of the baryon vertex to a state containing a single wrapped threebrane. In the final state, the N external charges are connected to each other or to the threebrane by elementary strings.

4.5. Instantons in AdS space

We will here conclude with a brief observations about the one Type IIB brane that we have so far overlooked – the -1-brane. It has already been noted [17] that -1-branes should be identified with instantons of the boundary conformal field theory. Here we will note an interesting fact relevant to this identification.

Consider the moduli space of $SU(2)$ instantons on $\mathbf{S}^4$ of instanton number one. Any such instanton is invariant under an $SO(5)$ subgroup of the conformal group $SO(5,1)$ of $\mathbf{S}^4$ [33]. Any two such instantons are related by an $SO(5,1)$ transformation. The moduli space of such instantons is hence a copy of $SO(5,1)/SO(5)$, that is a copy of AdS_5. By contrast, the moduli space for a single -1-brane on $AdS_5 \times \mathbf{S}^5$ is, of course, just a copy of $AdS_5 \times \mathbf{S}^5$. Clearly, a similar statement holds if $\mathbf{S}^5$ is replaced by $\mathbf{RP}^5/\mathbf{Z}_2$. It is tempting to identity the AdS_5 moduli space of the instanton with the first factor in the moduli space of the -1-brane; the relation between them hopefully comes by somehow averaging over the -1-brane position on $\mathbf{S}^5$ or $\mathbf{RP}^5$.

To consider the k-instanton moduli space, one should begin with gauge group $SU(N)$ for some large N. One component of the k-instanton moduli space is described by placing the k instantons in k commuting factors of $SU(2)$. This component is a symmetric product of k copies of AdS_5. The moduli space of the same number of -1-branes is meanwhile a symmetric product of k copies of $AdS_5 \times \mathbf{S}^5$ or $AdS_5 \times \mathbf{RP}^5$, obviously a closely related answer. The k-instanton moduli space also has other components, for instance with all k instantons in a common $SU(2)$. Perhaps these components make nonleading contributions for large N, in which case they might be difficult to see in the AdS description.

I should note in conclusion that a brane wrapping mode that has *not* been interpreted in the present paper is the twobrane made by wrapping a threebrane on $\mathbf{RP}^1 \subset \mathbf{RP}^5$. It would be interesting to know its interpretation in the boundary conformal field theory.

Acknowledgments

This work was supported in part by NSF Grant PHY-9513835. I would like to thank J. Blum, D. Freed, and N. Seiberg for discussions and comments.

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